Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Math. Pures Appl. 84 (2005) 1295–1361

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES

www.elsevier.com/locate/matpur

Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds

Xiang-Dong Li

Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118, route de Narbonne, 31062, Toulouse Cedex 4, France

Received 20 December 2004

Available online 14 July 2005

Dedicated to my daughter Anna Siyu for her fourth birthday

Abstract

Let $L=\Delta-\nabla\phi\cdot\nabla$ be a symmetric diffusion operator with an invariant measure $\mu(\mathrm{d}x)=\mathrm{e}^{-\phi(x)}\,\mathrm{d}x$ on a complete non-compact Riemannian manifold M. We give the optimal conditions on "the m-dimensional Ricci curvature associated with L" so that various Liouville theorems hold for L-harmonic functions, and that the heat semigroup $P_t=\mathrm{e}^{tL}$ has the C_0-diffusion property and is unique in $L^1(M,\mu)$. As applications, we give the optimal conditions for the uniqueness of the positive L-invariant measure and the L^1-uniqueness of the intrinsic Schrödinger operators on complete non-compact Riemannian manifolds. We also give a criterion for the finiteness of the total mass of the L-invariant measure and establish the Calabi–Yau volume growth theorem for the L-invariant measure on complete Riemannian manifolds on which "the m-dimensional Ricci curvature associated with L" is non-negative. This leads us to prove that if M is a complete Riemannian manifold with a finite L-invariant measure for which the associated m-dimensional Ricci curvature is non-negative, then M is compact. Moreover, we obtain an upper bound diameter estimate of such Riemannian manifolds by using the dimension of L, the total μ-volume of M and the upper bound of the μ-volume of geodesic balls of a fixed radius. Finally, using the variational formulae in Riemannian geometry, we give a new proof of the Bakry–Qian generalized Laplacian comparison theorem.

E-mail address: xiang@math.ups-tlse.fr (X.-D. Li).

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2005.04.002

Résumé

Soit $L = \Delta - \nabla\phi \cdot \nabla$ un opérateur de diffusion symétrique par rapport à une mesure invariante $\mu(\mathrm{d}x) = \mathrm{e}^{-\phi(x)}\,\mathrm{d}x$ sur une variété riemannienne complète non-compacte M. Dans cet article, nous donnons les conditions optimales sur « la courbure de Ricci de dimension m associée à L » pour établir les théorèmes de Liouville pour les fonctions L-harmoniques, la propriété de C_0-diffusion et l'unicité dans $L^1(M,\mu)$ pour le semigroupe de la chaleur $P_t = \mathrm{e}^{tL}$. Comme application, nous établissons l'unicité de la mesure L-invariante positive et l'unicité dans L^1 pour les opérateurs de Schrödinger intrinsèques sur les variétés riemanniennes complètes non-compactes. Nous donnons aussi un critère pour la finitude de la masse totale de la mesure L-invariante et établissons le théorème de la croissance de volume de Calabi–Yau pour la mesure L-invariante sur les variétés riemanniennes complètes pour lesquelles « la courbure de Ricci de dimension m associée à L » est non-négative. Ceci implique que si M est une variété riemannienne complète sur laquelle il existe un opérateur de diffusion symétrique L pour lequel la courbure de Ricci de dimension m associée est non-négative, et si la masse totale de la mesure L-invariante est finie, alors M est compacte. De plus, nous obtenons une estimation de la borne supérieure du diamètre de telles variétés par la dimension de L, le μ-volume total de M et la borne supérieure du μ-volume des boules géodésiques de rayon fixé. Enfin, en utilisant les formules variationnelles de géométrie riemannienne, nous donnons une nouvelle démonstration du théorème de comparaison de Bakry–Qian sur les laplaciens généralisés.

MSC: 58J05; 58J35; 58J65; 60J45; 60J60; 60H30

Keywords: Bakry–Emery Ricci curvature; C_0-diffusion property; L^1-uniqueness; Liouville theorems

1. Introduction and main results

1.1. Background

The strong (or weak) Liouville theorem in classical analysis states that every non-negative (or bounded) harmonic function on $\mathbb{R}^n$ must be constant. Since the middle of the seventies of the last century, Liouville theorems have been extensively studied on complete non-compact Riemannian manifolds, cf. [29,17]. In 1975, S.T. Yau [72] proved the L^∞-strong (or weak) Liouville theorem which says that if M is a complete Riemannian manifold with non-negative Ricci curvature then every positive (or bounded) harmonic function on M must be constant. In 1976, Yau [73] proved that on any complete non-compact Riemannian manifold M (without any other assumption) there are no non-negative L^p-subharmonic functions and no L^p-harmonic functions for any $1 < p < \infty$.

To prove the L^∞-Liouville theorem, Yau [72] developed the method of gradient estimate on complete Riemannian manifolds and proved that if M is a complete Riemannian manifold with Ricci curvature bounded from below by $-K$, i.e., $Ric \geqslant -K$, where $K \geqslant 0$ is a constant, then every harmonic function (i.e., $\Delta u = 0$) which is bounded from below satisfies the following gradient estimate

$$|\nabla u| \leqslant \sqrt{(n-1)K}\,\Big(u - \inf_M u\Big). \tag{1.1}$$

Letting $K = 0$, the gradient estimate (1.1) implies immediately Yau's L^∞-Liouville theorem. Another useful approach to prove the L^∞-Liouville theorems is to use the elliptic Harnack inequality or the (more strong) parabolic Harnack inequality. By independent works due to A. Grigor'yan [32] and L. Saloff-Coste [60], see also [61], the latter is equivalent to the doubling volume property and a family of the local Poincaré inequalities on geodesic balls. This implies that, if $(\widetilde{M}, \tilde{g})$ is a complete Riemannian manifold quasi-isometric to a complete Riemannian manifold (M, g) with non-negative Ricci curvature, then the L^∞-Liouville theorems hold on $(\widetilde{M}, \tilde{g})$. Let us mention that, due to the significant works of J.-J. Prat [56] (in two-dimensional case, 1975), D. Sullivan [67] (1983) and M. Anderson [2] (1983) on the existence of positive or bounded harmonic functions on Cartan–Hadamard manifolds with sectional curvature pinched between two negative constants, the L^∞-Liouville theorem does not hold on such manifolds, see also Anderson and Schoen [3], Ancona [1], Schoen and Yau [63], Hsu and March [34] and Hsu [35, 36]. Moreover T. Lyons [49] (1987) constructed two quasi-isometric complete Riemannian manifolds by which he proved that the validity of the L^∞-Liouville property is not stable under general quasi-isometric changes of Riemannian metrics.

The L^1-Liouville theorem has been studied by Garnett [27] (1983), P. Li and R. Schoen [42] (1983) and P. Li [41] (1984). Garnett [27] showed that if M is complete and has bounded geometry then M satisfies the L^1-Liouville theorem. Li and Schoen [42] proved that if the Ricci curvature on a complete Riemannian manifold satisfies

$$Ric(x) \geqslant -C\big(1+\rho^2(x)\big)\big(\log(1+\rho^2(x)\big)^{-\alpha}, \quad \forall x \in M, \tag{1.2}$$

where $C > 0$, $\alpha > 0$ are two constants, $\rho(x) = d(x, o)$ is the distance between $x \in M$ and a fixed point $o \in M$, then every non-negative L^1-integrable subharmonic function and every L^1-integrable harmonic function must be constant. The optimal geometric condition for the L^1-Liouville theorem was conjectured in Li and Schoen [42] and was affirmatively proved later in Li [41]: if the Ricci curvature is bounded below by a negative quadratic polynomial of the distance from a fixed point, that is, if there exists a constant $C > 0$ such that

$$Ric(x) \geqslant -C\big(1+\rho^2(x)\big), \quad \forall x \in M, \tag{1.3}$$

then the L^1-Liouville theorem holds. Under the same condition, Li also proved that every L^1-integrable solution of the heat equation $\partial_t u = \Delta u$ is unique determined by its initial date $u(\cdot, 0) \in L^1(M, \mathrm{d}x)$. That is to say, the Cauchy problem of the heat equation $\partial_t u = \Delta u$ is well-posed in $L^1(M, \mathrm{d}x)$ providing (1.3). For examples of complete Riemannian manifolds on which the L^1-Liouville theorems do not hold, we refer the reader to Chung [18], Li and Schoen [42] and Grigor'yan [31].

Two other versions of Liouville theorems have also been well studied in the literature. The first one is the strong Liouville property for the solution of $(\Delta - \lambda)u = 0$ for sufficiently large $\lambda > 0$. More precisely, this means that there exists some $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, every non-negative bounded solution of $(\Delta - \lambda)u = 0$ must be identically zero. This property is equivalent to the so-called stochastic completeness (i.e., the conservativeness) of Brownian motion on M in the probabilistic literature, see Davies [21], Grigor'yan [33] and Sturm [66]. By Yau [73], every complete Riemannian manifold with Ricci curvature bounded from below is stochastically complete. Karp and Li [39] proved that if the volume of the geodesic balls $B(x,R)$ of a complete Riemannian manifold M satisfies $V(B(x,r)) \leqslant e^{Cr^2}$ for some (and hence for all) $x \in M$ and all $r > 0$, then M is stochastically complete. By this criterion and the Bishop volume comparison theorem, Li [41] proved that if the Ricci curvature satisfies (1.3) then M is stochastically complete. Li's result can be also considered as a special case of a conservativeness criterion due to Varopoulos [68] and Hsu [36], where they proved that if there exists a positive increasing continuous function $K(r)$ on $[0,\infty)$ such that

$$\inf\bigl\{Ric(x)\colon\ \rho(x) = r\bigr\} \geqslant -(n-1)K(r) \tag{1.4}$$

and

$$\int_1^\infty \frac{\mathrm{d}r}{\sqrt{K(r)}} = \infty, \tag{1.5}$$

then M is stochastically complete.[1] So far it is well-known that the optimal geometric condition for the stochastic completeness of a complete Riemannian manifold is due to Grigor'yan [30] in which it was proved that if the volume of geodesic balls of a complete Riemannian manifold M satisfies

$$\int_1^\infty \frac{r\,\mathrm{d}r}{\log V(B(x,r))} = +\infty$$

for some (and hence for all) $x \in M$, then M is stochastically complete. The first example of complete but not stochastically complete Riemannian manifold was constructed by Azencott [5]. Lyons [50] showed that the stochastic completeness is not stable under general quasi-isometric changes of Riemannian metrics.

The second variant version of Liouville theorems is the so-called C_0-diffusion property or the Feller property of the Laplace–Beltrami operator on Riemannian manifolds. Let $C_0(M)$ be the space of continuous functions on M vanishing at infinity. We say that the Laplace–Beltrami operator on M has the C_0-diffusion property (or the Feller property) if $C_0(M)$ is stable under the heat semigroup $P_t = e^{t\Delta}$ for all $t > 0$. By Azencott [5], Δ has the C_0-property if and only if the following Liouville theorem holds for solutions of $(\Delta - \lambda)u = 0$ in the exterior region: for any compact subset $K \subset M$ and any $\lambda > 0$, the minimal positive solution of $(\Delta - \lambda)u = 0$ on $M \setminus K$ with the Dirichlet boundary condition $u \equiv 1$ on ∂K must tend to zero at infinity. If $\{W_t,\ t \geqslant 0\}$ denotes the Brownian motion on

[1] In his report on the first submitted version of this paper, Professor P. Malliavin pointed out to the author that a combination of Debiard, Gaveau and Mazet [23] and Vauthier [70] can give an immediate proof of the result mentioned here.

M starting at $x \in M$, then M has the C_0-property if and only if for each $t > 0$ and for all compact subset $K \subset M$,

$$\lim_{x \to \infty} P_x(T_K < t) = 0, \tag{1.6}$$

where $T_K = \inf\{t > 0\colon\ W_t \in K\}$ is the entrance time of Brownian motion $\{W_t,\ t \geqslant 0\}$ in K. Intuitively, the Laplace–Beltrami operator on M has the C_0-diffusion property if and only if the probability of Brownian motion starting from very far (near infinity) in M to visit any fixed compact subset before any fixed time is very small. In [74], Yau proved that every complete Riemannian manifold with Ricci curvature bounded from below by a negative constant has the C_0-property. We refer the reader to Dodziuk [24] for an alternative proof of this result for which one need only to use the maximum principle. By developing Azencott's idea, Hsu [35] proved that if M is a complete Riemannian manifold on which there exists a positive increasing continuous function $K(r)$ on $[0, \infty)$ satisfying (1.4) and (1.5), then the Laplace–Beltrami operator on M has the C_0-diffusion property.

1.2. Problems

The purpose of this paper is to study various Liouville theorems for general symmetric diffusion operators on complete Riemannian manifolds. Let M be a complete non-compact Riemannian manifold, ∇ be the gradient operator on M, Δ be the negative Laplace–Beltrami operator on M, ϕ be a C^∞ (or C^2) function on M. Consider the diffusion operator

$$L = \Delta - \nabla\phi \cdot \nabla$$

which is the infinitesimal generator of the Dirichlet form

$$\mathcal{E}(f, g) = \int_M (\nabla f, \nabla g)\,\mathrm{d}\mu, \quad \forall f, g \in C_0^\infty(M),$$

where μ is an invariant measure of L given by

$$\mathrm{d}\mu(x) = \mathrm{e}^{-\phi(x)}\,\mathrm{d}x.$$

By the Dirichlet form theory or Itô's SDE theory, it is well known that there exists a minimal diffusion processes $(X_t,\ t < \xi)$ on M whose infinitesimal generator is L, where ξ denotes the lifetime of X_t. In fact, X_t can be defined as the solution to the following Itô SDE:

$$\mathrm{d}X_t = \sqrt{2}\,\mathrm{d}W_t - \nabla\phi(X_t)\,\mathrm{d}t$$

where W_t denotes the Riemannian Brownian motion on M. The heat semigroup $P_t = \mathrm{e}^{tL}$ generated by L can be defined by:

$$P_t f(x) = E_x\big[f(X_t) 1_{[t<\xi]}\big], \quad \forall f \in C_0^\infty(M).$$

Clearly, $P_t = \mathrm{e}^{tL}$ is sub-Markovian, that is, $0 \leqslant f \leqslant 1$ implies $0 \leqslant P_t f \leqslant 1$ for all $t > 0$.

Now it is very natural to pose the following fundamental problems:[2]

Problem 1.1. What is the optimal geometric and analytic condition on M and ϕ such that the L^∞-Liouville theorems holds?

Problem 1.2. What is the optimal geometric and analytic condition on M and ϕ such that the heat semigroup $P_t = \mathrm{e}^{tL}$ is Markovian, i.e., $P_t 1 = 1$ for all $t > 0$, or equivalently, $(X_t,\ t < \xi)$ is conservative, i.e., $\xi \equiv \infty$?

Problem 1.3. What is the optimal geometric and analytic condition on M and ϕ such that $P_t = \mathrm{e}^{tL}$ has the C_0-diffusion property, i.e., P_t leaves $C_0(M)$ (the space of continuous functions vanishing at infinite) invariant?

Problem 1.4. What is the optimal geometric and analytic condition on M and ϕ such that the $L^1(\mu)$-Liouville theorem holds for L-harmonic functions and the solution in $L^1(\mu)$ of the heat equation $\partial_t u = Lu$ is uniquely determined by its initial date $u(\cdot, 0) \in L^1(\mu)$?

1.3. Motivation

The problems of establishing various Liouville theorems for the symmetric diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ are naturally arising in potential theory, probability theory and harmonic analysis on complete non-compact Riemannian manifolds. Moreover, the study of these problems has very important applications in other topics, for example, in the studies of the uniqueness of L-invariant measure and the Strichartz problem of the L^p-boundedness of the Riesz transform for ultracontractive symmetric diffusion operator on complete Riemannian manifolds. See Section 8.

A very important motivation to study the above problems can be illustrated by a deep connection between symmetric diffusion operators and Schrödinger operators naturally raised in quantum mechanics and quantum field theory. Here we borrow the ideas which have been extensively used in Davies [21], L.-M. Wu [71] and the reference therein. In fact, it is well-known that the diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ (called Nelson's diffusion operator in stochastic mechanics, see [71]), considered as a symmetric operator on $L^2(M, \mu)$, is unitary equivalent to the Schrödinger operator $H = \Delta - V$ with $V := \frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi$ (H is called the intrinsic Schrödinger operator in [21]), considered on $L^2(M, \mathrm{d}x)$. This unitary isomorphism $U : L^2(M, \mu) \to L^2(M, \mathrm{d}x)$ is given by

$$Uf := \mathrm{e}^{-\phi/2} f.$$

Under this unitary isomorphism, it is easy to see that u is a solution of $Lu = 0$ if and only if $v = \mathrm{e}^{-\phi/2}u$ is a solution of $Hv = 0$. Therefore, to say that every positive, or

[2] In his report on the first submitted version of this paper, Professor P. Malliavin informed the author that these problems have been studied by Vauthier [70] on complete Riemannian manifolds with bounded geometry.

bounded, or $L^1(\mu)$-integrable L-harmonic function u must be constant is equivalent to say that every positive, or $e^{-\phi/2}$-bounded (i.e., $|v(x)| \leqslant Ce^{-\phi(x)/2}$ for some constant $C \geqslant 0$ and for all $x \in M$), or $L^1(e^{-\phi/2}\,dx)$-integrable solution v of $Hv = 0$ must be of the form $v = Ce^{-\phi/2}$. Moreover, u is a solution to $(L-\lambda)u = 0$ if and only if $v = e^{-\phi/2}u$ is a solution to $(H-\lambda)v = 0$. Hence, to say that $P_t = e^{tL}$ is conservative (i.e., all the non-negative bounded solutions of $(L-\lambda)u = 0$ must be identically zero for sufficient large $\lambda > 0$) is equivalent to say that all the non-negative $e^{-\phi/2}$-bounded solutions v of $(H-\lambda)v = 0$ must be identically zero for sufficient large $\lambda > 0$.

Recall the correspondence between the L^p-uniqueness of e^{tL} and the one of the Schrödinger semigroup $Q_t = e^{tH}$ generated by H. The Feynman–Kac formula yields that for all $t \geqslant 0$,

$$U \circ P_t \circ U^{-1} = Q_t,$$

where

$$Q_t f(x) = E_x\left[f(W_t)\exp\left\{ -\int_0^t \left(\frac{|\nabla\phi(W_t)|^2}{4} - \frac{\Delta\phi(W_s)}{2} \right) ds \right\} 1_{[t<\chi]} \right], \quad \forall f \in C_0^\infty(M).$$

Here $\{W_t,\ t<\chi\}$ denotes the Riemannian Brownian motion on M starting at x with lifetime χ. Hence, to say that the heat semigroup $P_t = e^{tL}$ is $L^p(M,\mu)$-unique (i.e., the Cauchy problem of the heat equation $\partial_t u = Lu$ is well-posed in $L^p(M,\mu)$) with $\mu = e^{-\phi}\,dx$ is equivalent to say that the Schrödinger semigroup $Q_t = e^{tH}$ is $L^p(M, e^{\frac{p-2}{2}\phi}\,dx)$-unique (i.e., the Cauchy problem of the Schrödinger equation $\partial_t u = Hu$ is well-posed in $L^p(M, e^{\frac{p-2}{2}\phi}\,dx)$), where $p \in [1,\infty)$. In particular, $P_t = e^{tL}$ is $L^1(M, e^{-\phi(x)}\,dx)$-unique if and only if $Q_t = e^{tH}$ is $L^1(M, e^{-\phi(x)/2}\,dx)$-unique. Moreover, it is easy to see that $P_t = e^{tL}$ has the C_0-diffusion property is equivalent to say that

$$C_\phi(M) := \left\{ g \in C(M)\colon \lim_{d(x,o)\to 0} e^{\phi(x)/2} f(x) = 0 \right\}$$

is stable under the Schrödinger semigroup $Q_t = e^{tH}$ for all $t > 0$.

Due to the above correspondence between the properties of the symmetric diffusion operator $L = \Delta - \nabla\phi\cdot\nabla$ and the ones of the Schrödinger operator $H = \Delta - (|\nabla\phi|^2/4 - \Delta\phi/2)$, we can therefore transfer the conditions required for the positive answers of Problems 1.1 to 1.4 to the corresponding problems for H, which are of fundamental importance in quantum mechanics and quantum field theory.

1.4. Method

It is helpful to point out that it was Yau who first realized in his seminal papers [72–74] that one can combine the PDEs approach (mainly the maximum principle) with some geometric techniques including the use of the Bochner–Weitzenböck formula and various comparison theorems in Riemannian geometry to deal with the problem of Liouville theorems on complete non-compact Riemannian manifolds. While on the other hand, the stochastic method in geometric analysis has been systematically developed by many people (e.g., [5,20,23,25,34–37,49,50,56,67–70]) since the pioneering work of P. Malliavin [51–53] in 1974–1975. In particular, Bakry and his collaborators [10,6–9,12,13,11] have developed a successful method to deal with analytic and geometric problems related to diffusion operators on Riemannian manifolds based on the introduction of the "Ricci curvature of a diffusion operator" and the use of the "curvature-dimension inequality". In this paper we will use a combination of these methods to establish various Liouville theorems for diffusion operators on complete Riemannian manifolds.

1.5. Bakry–Emery Ricci curvature

To develop the rest part of this paper, let us introduce the most important notion, "the m-dimensional Ricci curvature associated with a diffusion operator", which plays a crucial role in the study of our problems. For any constant $m \geqslant n = \dim M$, we introduce the symmetric 2-tensor

$$Ric_{m,n}(L)(x) := Ric(x) + \nabla^2\phi(x) - \frac{\nabla\phi(x)\otimes\nabla\phi(x)}{m-n}, \quad \forall x \in M, \tag{1.7}$$

and call it the m-dimensional Bakry–Emery Ricci curvature of the diffusion operator $L = \Delta - \nabla\phi\cdot\nabla$. Throughout this paper we use the convention that $m = n$ if and only if $L = \Delta$.

Following Bakry [8], we say that L satisfies the curvature-dimension $CD(K,m)$ condition (or the curvature-dimension $CD(K,m)$ inequality) if

$$\Gamma_2(u,u) \geqslant \frac{1}{m}(Lu)^2 + K|\nabla u|^2, \quad \forall u \in C^\infty(M),$$

where

$$\Gamma_2(u,u) := \frac{1}{2}L|\nabla u|^2 - \langle\nabla Lu, \nabla u\rangle.$$

By Bakry [6,8], the generalized Bochner–Weitzenböck formula holds

$$\Gamma_2(u,u) = |\nabla^2 u|^2 + \big\langle Ric(L)\nabla u, \nabla u\big\rangle,$$

where $Ric(L)$ is the ∞-dimensional Bakry–Emery Ricci curvature defined by:

$$Ric(L) = Ric + \nabla^2\phi.$$

Since

$$|\nabla^2 u|^2 \geqslant \frac{1}{n}|\Delta u|^2,$$

it follows that the $CD(K,n)$ condition holds when L is the Laplace–Beltrami operator Δ on a complete Riemannian manifold with $Ric \geqslant K$. Moreover, using the elementary inequality:

$$(a+b)^2 \geqslant \frac{a^2}{1+\alpha} - \frac{b^2}{\alpha}, \quad \forall \alpha > 0,$$

and by identifying $\nabla\phi \otimes \nabla\phi(\nabla u, \nabla u)$ with $|\nabla\phi \cdot \nabla u|^2$, we obtain

$$\begin{aligned} \Gamma_2(u,u) &\geqslant \frac{1}{n}|\Delta u|^2 + \langle Ric(L)\nabla u, \nabla u\rangle \\ &= \frac{1}{n}|Lu + \nabla\phi \cdot \nabla u|^2 + \langle Ric(L)\nabla u, \nabla u\rangle \\ &\geqslant \frac{1}{n(1+\alpha)}|Lu|^2 - \frac{\nabla\phi \otimes \nabla\phi(\nabla u, \nabla u)}{n\alpha} + \langle Ric(L)\nabla u, \nabla u\rangle. \end{aligned}$$

Hence

$$\Gamma_2(u,u) \geqslant \frac{1}{n(1+\alpha)}|Lu|^2 + \left(Ric(L) - \frac{\nabla\phi \otimes \nabla\phi}{n\alpha}\right)(\nabla u, \nabla).$$

Setting

$$m := (1+\alpha)n,$$

then

$$\Gamma_2(u,u) \geqslant \frac{1}{m}|Lu|^2 + \left(Ric(L) - \frac{\nabla\phi \otimes \nabla\phi}{m-n}\right)(\nabla u, \nabla u).$$

Therefore, we find that any diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ on any n-dimensional complete Riemannian manifold (M,g) satisfies the $CD(K_{m,n}, m)$ condition for all $m > n$, where

$$K_{m,n}(x) = \inf\left\{Ric(V,V) + \nabla^2\phi(V,V) - \frac{|\nabla\phi(V)|^2}{m-n}\colon\ V \in T_xM,\ \|V\| = 1\right\}.$$

This also explains why and how the quantity $Ric_{m,n}(L)$ in (1.7) was introduced.

Remark 1.1. The Bakry–Emery Ricci curvature was introduced when Bakry and Emery [10] studied the logarithmic Sobolev inequality for diffusion operators on a complete Riemannian manifold. It plays a crucial role in the study of the L^p-boundedness of Riesz transforms associated with diffusion operators on complete Riemannian manifolds (cf. Bakry [7] and Li [44]). In a recent paper by G. Perelman [54], it has been used to modify R. Hamilton's Ricci flow towards the proof of Poincaré's conjecture of 3-dimensional manifold. Recently, Lott [47] gave a new understanding of the Bakry–Emery Ricci curvature by using the warped product metric. Suppose that $q = m - n \in \mathbb{N}$. Given $k \in \mathbb{Z}^+$, consider $S^q \times M$ with the warped product metric defined by

$$g_k^{S^q \times M} = g^M + k^{-2} e^{-\frac{2\phi}{q}} g^{S^q}.$$

Let $\pi : S^q \times M \to M$ be the natural projection map. Let $\overline{X}$ be the horizontal lift of a vector field X on M to $S^q \times M$. Then

$$Ric^{S^q \times M}\big(\overline{X}, \overline{X}\big) = \pi^*\big(Ric^M(X, X) - q e^{\phi/q} \nabla^2 e^{-\phi/q}(X, X)\big).$$

Equivalently,

$$Ric_{m,n}(X, X) := Ric^M(X, X) + \nabla^2\phi - \frac{1}{q}\nabla\phi \otimes \nabla\phi(X, X) = \pi_*\big(Ric^{S^q \times M}\big(\overline{X}, \overline{X}\big)\big).$$

In [47], Lott pointed out that, if $Ric_q \geqslant Kg$, then as $k \to \infty$, (M, g, ϕ) is the Hausdorff limit of a sequence of $(m = n + q)$-dimensional manifolds $(S^q \times M, g_k^{S^q \times M})$ with Ricci curvature bounded from below by K.

1.6. The Bakry–Qian comparison theorem

The following result, which extends the well-known Myers theorem and the Laplacian comparison theorem in Riemannian geometry, is due to Bakry–Qian [13]. In Section 10, we give a new proof of this theorem which plays an important role in this paper.

Theorem 1.1 (Bakry and Qian [13]). *Let M be a complete Riemannian manifold, L be a diffusion operator satisfying the curvature-dimension condition $CD(K, m)$, where $K = K(\rho(x))$ is a function depending on $\rho(x) = d(x, o)$, $o \in M$ is a fixed point. Let a_K be the solution defined on the maximal interval $(0, \delta_K)$ of the following Riccati equation*

$$-a_K'(\rho) = K + \frac{a_K^2(\rho)}{m - 1}$$

with the boundary condition $\lim_{x\to 0} x a_K(x) = m - 1$. Here δ_K is the explosion time of a_K such that

$$\lim_{x\to\delta_K -} a_K(x) = -\infty.$$

Then

(1) *If $\delta_K < \infty$, then the extended Myers theorem holds, that is, M is compact and the diameter of M satisfies*

$$D(M) \leqslant \delta_K.$$

(2) *The generalized Laplacian comparison theorem holds, that is,*

$$L\rho(x) \leqslant a_K\big(\rho(x)\big), \quad \forall x \in M \setminus cut(o).$$

Corollary 1.2 [57,58,47,13]. *Let L be a diffusion operator satisfying the curvature-dimension condition $CD(-K, m)$, where $K > 0$ is a constant. Then*

$$L\rho \leqslant (m-1)\sqrt{K}\,\rho\coth\big(\sqrt{K}\rho\big), \quad \forall x \in M \setminus cut(o).$$

Under the curvature-dimension condition $CD(K, m)$, the Bishop volume comparison theorem follows from the generalized Laplacian comparison theorem. Moreover, the Cheeger–Gromov–Taylor relative volume comparison theorem also holds, see, e.g., [57, 58,28,47,13].

1.7. Main results

Based on the Bakry–Qian generalized Laplacian comparison theorem, using an interplay between the methods of geometric analysis and stochastic analysis, and by some new ingredients that we will explain later, we are able to extend the previous results of Yau, Li, Schoen, Azencott, Hsu and Varopoulos for the Laplace–Beltrami operator to general symmetric diffusion operators under the optimal geometric and analytic conditions. Here the meaning of "the optimal geometric and analytic conditions" should be understood as the weakest possible conditions only involving the Ricci curvature of M and the derivatives of ϕ. That is, we do not use other geometric or analytic quantities (such as the spectral gap or the weight volume of geodesic balls, etc.) to describe the optimal condition.

We now state the main results of this paper as follows:[3]

Theorem 1.3. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exist two constants $m \geqslant n$ and $K \geqslant 0$ such that*

$$Ric_{m,n}(L)(x) \geqslant -K, \quad \forall x \in M. \tag{1.8}$$

Then for any solution of $Lu = 0$ which is bounded from below, we have

$$|\nabla u| \leqslant \sqrt{(m-1)K}\,\Big(u - \inf_M u\Big). \tag{1.9}$$

In particular, if $Ric_{m,n}(L) \geqslant 0$, then every positive or bounded solution to $Lu = 0$ must be constant.

[3] Recall that throughout this paper, $Ric_{m,n}(L)$ is well-defined for all $m > n$ if $L = \Delta - \nabla\phi \cdot \nabla$ is a diffusion operator with a non-vanishing drift term $\nabla\phi \cdot \nabla$, while we use the convention that $m = n$ if and only if $L = \Delta$ (in this case $Ric_{n,n}(\Delta) = Ric$).

Theorem 1.4. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exist a point $o \in M$ and a positive increasing continuous function $K(r)$ on $(0,\infty)$ such that*

$$Ric(x) + \nabla^2\phi(x) - \frac{\nabla\phi(x) \otimes \nabla\phi(x)}{m-n} \geqslant -K\big(\rho(x)\big), \quad \forall x \in M,$$

where $\rho(x) = d(x,o)$, and

$$\int_1^\infty \frac{\mathrm{d}r}{\sqrt{K(r)}} = +\infty.$$

Then the heat semigroup $P_t = \mathrm{e}^{tL}$ is conservative, i.e.,

$$P_t 1(x) = 1, \quad \forall t > 0,\ x \in M.$$

Equivalently, there exists some $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, every non-negative bounded solution of $(L-\lambda)u = 0$ must be identically zero.

Theorem 1.5. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exist a constant $m > n$ and a positive increasing continuous function $K(r)$ on $[0,\infty)$ such that*

$$\inf\big\{Ric_{m,n}(x):\ \rho(x) = r\big\} \geqslant -K(r)$$

with

$$\int_1^\infty \frac{\mathrm{d}r}{\sqrt{K(r)}} = \infty.$$

Then the heat semigroup $P_t = \mathrm{e}^{tL}$ has the C_0-property or the Feller property, i.e., $C_0(M)$ is stable under $P_t = \mathrm{e}^{tL}$ for all $t > 0$. Equivalently, for any $\lambda > 0$ and any compact subset $K \subset M$, the minimal positive solution of $(L-\lambda)u = 0$ on $M \setminus K$ with the Dirichlet boundary condition $u \equiv 1$ on ∂K must tend to zero at infinity.

Theorem 1.6. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exist two constants $m > n$ and $C > 0$ such that*

$$Ric_{m,n}(L)(x) \geqslant -C\big(1 + \rho^2(x)\big), \quad \forall x \in M. \tag{1.10}$$

Then every non-negative $L^1(\mu)$-integrable L-subharmonic function and every $L^1(\mu)$-integrable L-harmonic function on M must be constant. Moreover, every $L^1(\mu)$-integrable solution of the heat equation $\partial_t u = Lu$ is uniquely determined by its initial date in $L^1(\mu)$.

Applications of the above results will be given in Section 8. As an application of Theorem 1.3, we prove that if there exists a constant $m \geqslant n$ such that $Ric_{m,n}(x) \geqslant 0$ for all $x \in M$, then all the positive L-invariant measures must be of the form $c\mu$ where $c \in \mathbb{R}^+$, see Theorem 8.1. Combining Theorem 1.4 with a result due to the author on the L^p-boundedness of Riesz transforms for ultracontractive symmetric diffusion operators, see [44], we find a new class of symmetric diffusion operators on complete non-compact Riemannian manifolds for which the Bakry–Emery Ricci curvature is not necessarily to be uniformly bounded from below while the Riesz transforms are bounded in $L^p(\mu)$ for all $p \geqslant 2$, see Proposition 8.3.

After the first submitted version of this paper was accepted for publication on January 19, 2005, the author has obtained some new results which are given in Section 9. To save the length of the paper, we would not like to describe these results here. In Section 10, we give a new proof of the Bakry–Qian generalized Laplacian comparison theorem by using the variational formulae in Riemannian geometry.

1.8. Comparison with known results

Before to end this long Introduction, let us give some remarks on the comparison of our main theorems with some known results in the literature.

Remark 1.2. Theorem 1.3 extends Yau's gradient estimate (1.1) and his famous Liouville theorem for harmonic functions of the Laplace–Beltrami operator Δ to the general symmetric diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$. Even though the result is a very natural extension of Yau's theorem, it seems to the author that one cannot find it in the literature and its proof needs a careful estimate which will be given in Section 2 below.

Remark 1.3. In the case where $L = \Delta$, Theorem 1.4 was proved by Varopoulos [68] (1983) and Hsu [36] (1989). As pointed out by Professor P. Malliavin to the author, it can be obtained via a combination of Debiard, Gaveau and Mazet [23] (1976/1977) and Vauthier [70] (1979). We refer the reader to Grigor'yan [33] for a complete description of known results and references in the study of the stochastic completeness on complete Riemannian manifolds. There have been many known results on the conservativeness and the C_0-property of diffusion processes on non-compact manifolds in the literature. In [6], Bakry proved that if there exists a constant $K \geqslant 0$ such that $Ric(x) + \nabla^2\phi(x) \geqslant -K$ for all $x \in M$, then the L-diffusion process is conservative (in the case where $L = \Delta$, this result is due to Yau [72]). Elworthy [25] and Davies [22] have given some criteria for the conservativeness of diffusion operators. In the case where M is a non-compact Riemannian manifold with a pole o (this requires that $r(\cdot) = d(\cdot, o)$ is a smooth function on M), X.-M. Li [45] proved that, for diffusion operator $L = \Delta + Z$ (where Z is not necessarily to be of the form $Z = \nabla\phi$), the L-diffusion process is conservative if $Ric(x) \geqslant -C(1 + d^2(x, o))$ and $\mathrm{d}r(Z) \leqslant C(1 + d(x, o))$ for all $x \in M$. Theorem 1.4 is very closed to a result due to Zhongmin Qian (Indeed, the author proved Theorems 1.4 and 1.6 in June 2004 and was only aware of Qian's paper [57] and his results on the conservativeness and the C_0-diffusion property of diffusion operators (see Remark 1.4 below) in September 2004). It was proved in [57] that the heat semigroup generated by $L = \Delta + B$

is conservative provided that there exist two positive continuous nondecreasing functions $k_i : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\lim_{t\to+\infty} k_i(t) = +\infty$ and

$$\int_1^\infty \frac{1}{\sqrt{k_1^2(t)+k_2^2(t)}}\,\mathrm{d}t = +\infty,$$

and such that $(Ric - \nabla^S B)(x) \geqslant -k_1^2(d(x,o))$ and $|B|(x) \leqslant k_2(d(x,o))$ for all $x \in M$, where $\nabla^S B$ is the symmetric part of ∇B defined by

$$\nabla^S B(\xi,\eta) = \frac{1}{2}\big\{\langle\nabla_\xi B,\eta\rangle + \langle\nabla_\eta B,\xi\rangle\big\}, \quad \forall \xi,\eta \in TM.$$

In the case where $B = -\nabla\phi$, Qian's condition requires that $Ric(x) + \nabla^2\phi(x) \geqslant -k_1^2(d(x,o))$ and $|\nabla\phi(x)| \leqslant k_2(d(x,o))$ for all $x \in M$. Note that

$$Ric_{m,n} = Ric + \nabla^2\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}.$$

It is easy to see that if $L = \Delta - \nabla\phi\cdot\nabla$ satisfies Qian's condition, then

$$Ric_{m,n}(x) \geqslant -k_1^2\big(d(x,o)\big) - \frac{k_2^2(d(x,o))}{m-n}$$

for any constant $m > n$. Theorem 1.4 gives the optimal curvature-dimension condition (in the sense of Bakry and Emery) for the conservativeness of $P_t = \mathrm{e}^{tL}$ on finite dimensional complete Riemannian manifolds. However, we don't know whether or not Theorem 1.4 holds on infinite dimensional manifold by taking $m \to \infty$.

Remark 1.4. The concept of C_0-diffusion property was introduced by Azencott [5] and goes back to W. Feller [26]. Azencott [5] developed the connection between diffusion semigroup having the C_0-diffusion property and the Liouville property at infinity for harmonic functions solving $(L-\lambda)u = 0$ for some $\lambda > 0$ in the exterior region of compact sets $K \subset M$ and satisfying the Dirichlet boundary condition $u|_{\partial K} = 1$, see Section 1.1. In Section 8.4 and p. 433 of [55], R.G. Pinsky described Azencott's idea by "the explosion inward from the boundary". When L is an one-dimensional diffusion operator on open interval, sufficient and necessary conditions so that the C_0-property holds or does not hold are given in [5,55]. Azencott [5] proved the C_0-diffusion property for the Laplace–Beltrami operator on complete analytic simple connected Riemannian manifolds of negative curvature. Yau [74] and Dodziuk [24] proved the C_0-diffusion property for the Laplace–Beltrami operator on complete Riemannian manifolds with Ricci curvature bounded from below by a negative constant. Theorem 1.6 extends Hsu's result in [36] in which $L = \Delta$, see Section 1.1. Elworthy [25] and Davies [21] have given some criteria for the C_0-diffusion property of diffusion operators. Qian [57] proved that for $L = \Delta + B$ if $Ric - \nabla^S B \geqslant -k$ for some nonnegative constant k and $|B(x)| \leqslant C(1 + d(x,o))$

for all $x \in M$ and a fixed point $o \in M$, then $P_t = e^{tL}$ has the C_0-diffusion property. Clearly, the assumption required in our Theorem 1.5 is weaker than Qian's condition for $L = \Delta - \nabla\phi \cdot \nabla$.

Remark 1.5. The L^1-Liouville theorem for harmonic functions of the Laplace–Beltrami operator on complete non-compact Riemannian manifold is strongly relied on the behavior of curvature. Garnett [27] showed that if M is complete and has bounded geometry (i.e., the Riemannian curvature tensor and all of its higher order derivatives are bounded and the injectivity radius is strictly positive), then M satisfies the L^1-Liouville theorem. This result was improved by Li and Schoen [42] who proved that the L^1-Liouville theorem for the Laplace–Beltrami operator holds if the Ricci curvature on M is bounded from below and the volume of every geodesic ball in M has a positive lower bound. Li and Schoen [42] further proved the L^1-Liouville theorem under the condition (1.2) and conjectured that the optimal curvature condition for the L^1-Liouville theorem is (1.3). This was affirmatively proved by P. Li [41]. Example of complete or even stochastically complete Riemannian manifolds on which the L^1-Liouville theorem does not hold have been given in Li and Schoen [42] and Chung [18]. Theorem 1.6 extends P. Li's L^1-Liouville theorem and his result on the L^1-uniqueness of heat semigroup $e^{t\Delta}$ for the Laplace–Beltrami operator Δ on complete non-compact Riemannian manifolds satisfying the optimal curvature condition (1.3) to general symmetric diffusion operators satisfying the curvature-dimension condition (1.10). In the case where ϕ is not constant, Theorem 1.6 is new and should be considered as the most important result of this paper. Indeed, one of the original motivations for the author to begin the study of the problems in this paper was exactly trying to extend P. Li's results in [41] to general diffusion operators under a natural optimal curvature condition (in the sense of Bakry and Emery). The proof of Theorem 1.6 needs many careful heat kernel estimates and consists of the main part of the present paper.

Remark 1.6. In the case where $M = \mathbb{R}^n$ or $M = D$ is an open domain of $\mathbb{R}^n$, and $L = \Delta + 2\frac{\nabla\phi}{\phi} \cdot \nabla$ is a diffusion operator with singular drift term $\frac{\nabla\phi}{\phi}$, where $\phi \in H^{1,2}(\mathbb{R}^n, \mathrm{d}x)$, the problem of whether L has a unique extension in $L^1(M, \phi^2\,\mathrm{d}x)$ which generates a C_0-semigroup in $L^1(M, \phi^2 dx)$ has been studied by Liskevitch [46], Stannat [64] and L.-M. Wu [71]. In [71], Wu proved that $(L, C_0^\infty(D))$ is $L^1(D, \phi^2\,\mathrm{d}x)$-unique if and only if the corresponding diffusion is conservative under the condition that ϕ is continuous and $\phi \in H^1_{\mathrm{loc}}(D)$. The reader might wonder why the $L^1(D, \phi^2\,\mathrm{d}x)$-uniqueness of $L = \Delta - 2\frac{\nabla\phi}{\phi} \cdot \nabla$ is equivalent to the conservativeness of $P_t = e^{tL}$ since in the case where $D = \mathbb{R}^n$ and $\phi \in C^2(\mathbb{R}^n)$, it is clear that ϕ satisfies Wu's conditions and hence if the above statement is true then L is $L^1(\mathbb{R}^n, \phi^2\,\mathrm{d}x)$-unique if and only if the L-diffusion process is conservative. In November 2004, Professor Wu kindly pointed out to the author that, the concept of "the L^1-uniqueness" studied in [46,64,71] is exactly the dual concept of the L^1-uniqueness discussed here and in [27,42,41] and is therefore equivalent to the problem of conservativeness of the heat semigroup $P_t = e^{tL}$.

2. Gradient estimate and strong Liouville theorem

2.1. The key lemma

The key lemma used in the proof of Yau's gradient estimate (1.1), see Yau [72] or Schoen and Yau [63], is that under the curvature condition $Ric \geqslant -(n-1)K$, where $K \geqslant 0$, for any harmonic function u (i.e., $\Delta u = 0$), we have

$$|\nabla u|\Delta|\nabla u| + (n-1)K|\nabla u|^2 \geqslant \frac{1}{n-1}\big|\nabla\big(|\nabla u|\big)\big|^2. \tag{2.11}$$

This is a consequence of the classical Bochner–Wëitzenbock formula. We now extend Yau's lemma to any diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ under a suitable curvature-dimension condition in terms of $Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}$.

Lemma 2.1. *Let u be a solution of $Lu = 0$, $m > n$ be a constant. Then*

$$|\nabla u|L|\nabla u| \geqslant \frac{1}{m-1}\big|\nabla\big(|\nabla u|\big)\big|^2 + \big\langle Ric_{m,n}(L)\nabla u, \nabla u\big\rangle. \tag{2.12}$$

Proof. By the generalized Bochner–Weitzenböck formula, for any $u \in C^2(M)$ we have

$$\frac{1}{2}L|\nabla u|^2 = |\nabla^2 u|^2 + \big\langle Ric(L)\nabla u, \nabla u\big\rangle + \langle \nabla Lu, \nabla u\rangle.$$

By the diffusion property,

$$L|\nabla u|^2 = 2|\nabla u|L|\nabla u| + 2\big|\nabla\big(|\nabla u|\big)\big|^2.$$

Therefore, if u is a solution of $Lu = 0$, we have

$$|\nabla u|L|\nabla u| = |\nabla^2 u|^2 - \big|\nabla\big(|\nabla u|\big)\big|^2 + \big\langle Ric(L)\nabla u, \nabla u\big\rangle. \tag{2.13}$$

By Yau [72] and Schoen and Yau [63], if we consider a local normal chart at x in which $u_1(x) = |\nabla u|(x)$ and $u_i(x) = 0$ for all $i \geqslant 2$, then $\nabla_j|\nabla u| = u_{1j}$ and hence $|\nabla(|\nabla u|)|^2 = \sum_j u_{1j}^2$. Since u is a solution of $Lu = 0$, in the above local geodesic chart we have

$$\sum_{i=2}^{n} u_{ii} = -u_{11} + \phi_1 u_1.$$

Hence

$$
\begin{aligned}
|\nabla^2 u|^2 - \big|\nabla\big(|\nabla u|\big)\big|^2 &= \sum_{i,j} u_{ij}^2 - \sum_j u_{1j}^2 = \sum_{i\neq 1,\ j\geqslant 1} u_{ij}^2 \geqslant \sum_{i=2}^n u_{i1}^2 + \sum_{i=2}^n u_{ii}^2 \\
&\geqslant \sum_{i=2}^n u_{i1}^2 + \frac{1}{n-1}\left(\sum_{i=2}^n u_{ii}\right)^2 = \sum_{i=2}^n u_{i1}^2 + \frac{1}{n-1}\,(u_{11} + \phi_1 u_1)^2 .
\end{aligned}
$$

Using the elementary inequality

$$
(a+b)^2 \geqslant \frac{a^2}{1+\alpha} - \frac{b^2}{\alpha}, \quad \forall \alpha > 0,
$$

we obtain:

$$
(u_{11} + \phi_1 u_1)^2 \geqslant \frac{u_{11}^2}{1+\alpha} - \frac{|\phi_1 u_1|^2}{\alpha}, \quad \forall \alpha > 0.
$$

Hence, for all $\alpha > 0$, we have:

$$
\begin{aligned}
|\nabla^2 u|^2 - \big|\nabla\big(|\nabla u|\big)\big|^2 &\geqslant \sum_{i=2}^n u_{i1}^2 + \frac{1}{n-1}\left[\frac{u_{11}^2}{1+\alpha} - \frac{|\phi_1 u_1|^2}{\alpha}\right] \\
&= \left(\sum_{i=2}^n u_{i1}^2 + \frac{1}{(1+\alpha)(n-1)} u_{11}^2\right) - \frac{1}{\alpha(n-1)}|\phi_1 u_1|^2 \\
&\geqslant \frac{1}{(1+\alpha)(n-1)} \sum_{i=1}^n u_{i1}^2 - \frac{1}{\alpha(n-1)}|\phi_1 u_1|^2.
\end{aligned}
$$

Therefore, for all $\alpha > 0$,

$$
|\nabla^2 u|^2 - \big|\nabla\big(|\nabla u|\big)\big|^2 \geqslant \frac{1}{(1+\alpha)(n-1)}\big|\nabla\big(|\nabla u|\big)\big|^2 - \frac{1}{\alpha(n-1)}|\nabla\phi\cdot\nabla u|^2.
$$

Combining this with (2.13), for all $\alpha > 0$, we have

$$
\begin{aligned}
|\nabla u| L |\nabla u| &\geqslant \frac{1}{(1+\alpha)(n-1)}\big|\nabla\big(|\nabla u|\big)\big|^2 - \frac{1}{\alpha(n-1)}|\nabla\phi\cdot\nabla u|^2 + \big\langle Ric(L)\nabla u, \nabla u\big\rangle \\
&= \frac{1}{(1+\alpha)(n-1)}\big|\nabla\big(|\nabla u|\big)\big|^2 + \left\langle\left(Ric(L) - \frac{\nabla\phi\otimes\nabla\phi}{\alpha(n-1)}\right)\nabla u, \nabla u\right\rangle.
\end{aligned}
$$

Taking $\alpha = \frac{m-n}{n-1}$, then $(1+\alpha)(n-1) = m-1$. This yields the desired inequality. $\square$

2.2. Generalized Yau's gradient estimate

Theorem 2.2. *Suppose that $Ric_{m,n}(L) \geqslant -K$, where $K \geqslant 0$ is a constant. Let u be a solution of $Lu = 0$ which is bounded from below. Then*

$$|\nabla u| \leqslant \sqrt{(m-1)K}\left(u - \inf_M u\right). \tag{2.14}$$

In particular, if $Ric_{m,n}(L) \geqslant 0$, then every positive solution (*and bounded solution*) *of $Lu = 0$ must be constant.*

Proof. The proof is similar to the one in Yau [72] and Schoen and Yau [63]. Suppose u be a positive solution of $Lu = 0$. Let $\psi = |\nabla \log u|$. Then

$$\nabla \psi = \frac{\nabla |\nabla u|}{u} - \frac{|\nabla u| \nabla u}{u^2}. \tag{2.15}$$

At any point where $|\nabla u| \neq 0$, by the diffusion property, we have

$$L\big(|\nabla u|\big) = L(\psi u) = uL\psi + \psi Lu + 2\nabla\psi \cdot \nabla u = uL\psi + 2\nabla\psi \cdot \nabla u.$$

Using (2.12) we have

$$\begin{aligned} L\psi &= \frac{L|\nabla u|}{u} - \frac{2\nabla\psi \cdot \nabla u}{u} = \frac{|\nabla u| L(|\nabla u|)}{|\nabla u| u} - \frac{2\nabla\psi \cdot \nabla u}{u} \\ &\geqslant \frac{1}{|\nabla u| u}\left(\frac{1}{m-1}\big|\nabla\big(|\nabla u|\big)\big|^2 - K|\nabla u|^2\right) - \frac{2\nabla\psi \cdot \nabla u}{u} \\ &= \frac{1}{m-1}\frac{|\nabla(|\nabla u|)|^2}{|\nabla u| u} - K\psi - \frac{2\nabla\psi \cdot \nabla u}{u}. \end{aligned}$$

Let $\varepsilon = 2/(m-1)$. Using (2.15) we have (see Schoen and Yau [63])

$$\begin{aligned} \frac{2\nabla\psi \cdot \nabla u}{u} &= (2-\varepsilon)\frac{\nabla\psi \cdot \nabla u}{u} + \varepsilon\frac{\nabla(|\nabla u|) \cdot \nabla u}{u^2} - \varepsilon\frac{|\nabla u|^3}{u^3} \\ &\leqslant (2-\varepsilon)\frac{\nabla\psi \cdot \nabla u}{u} + \varepsilon\frac{|\nabla(|\nabla u|)||\nabla u|}{u^2} - \varepsilon\psi^3. \end{aligned}$$

By Schoen and Yau [63], we have

$$\begin{aligned} \varepsilon\frac{|\nabla(|\nabla u|)|\nabla u|}{u^2} &\leqslant \frac{\varepsilon}{2}\left(\frac{|\nabla(|\nabla u|)|^2}{|\nabla u| u} + \frac{|\nabla u|^3}{u^3}\right) \\ &= \frac{1}{m-1}\left(\frac{|\nabla(|\nabla u|)|^2}{|\nabla u| u} + \psi^3\right). \end{aligned}$$

From the above two inequalities we obtain

$$L\psi \geqslant -K\psi - \left(2 - \frac{2}{m-1}\right)\frac{\nabla\psi\cdot\nabla u}{u} + \frac{\psi^3}{m-1}. \tag{2.16}$$

Now suppose that ψ achieves its maximum at some point $x_0 \in M$. Then $\nabla\psi(x_0) = 0$ and $\Delta\psi(x_0) \leqslant 0$. Hence $L\psi(x_0) = \Delta\psi(x_0) - \nabla\phi(x_0)\cdot\nabla\psi(x_0) \leqslant 0$. By this and (2.16) we obtain

$$0 \geqslant -K\psi(x_0) + \frac{\psi^3(x_0)}{m-1}.$$

Therefore $\psi(x_0) \leqslant \sqrt{(m-1)K}$ and hence $\psi(x) \leqslant \sqrt{(m-1)K}$ holds for all $x \in M$. Hence

$$|\nabla u| \leqslant \sqrt{(m-1)K}\, u.$$

Replacing u by $u - \inf_M u$ we finish the proof of (2.14). The strong and the weak Liouville theorems follow easily from (2.14) by taking $K = 0$ when $Ric_{m,n}(L) \geqslant 0$. □

2.3. Schoen–Yau gradient estimate and Harnack inequality

The following result is a natural extension of a theorem due to Schoen and Yau [63].

Theorem 2.3. *Suppose that $Ric_{m,n}(L) \geqslant -K$, u be a positive solution of $\Delta u = 0$. Then there exists a constant C_n such that for all $a > 0$,*

$$\sup_{y\in B(x,a/2)} \frac{|\nabla u(y)|}{u(y)} \leqslant C_m\left(\frac{1}{a} + \sqrt{K}\right).$$

Proof. The proof is similar to the one of Theorem 3.1 in Schoen and Yau [63]. For any $a > 0$, let us define

$$F(y) = \big(a^2 - d^2(x,y)\big)\frac{|\nabla u|}{u}, \quad y \in B(x,a).$$

Since $F|_{\partial B(x,a)} = 0$, if $|\nabla u| \neq 0$, then F can only achieve its maximum at some point $x_0 \in B(x,a)$. Assume that $x_0 \notin cut(x)$, then F is smooth near x_0 and by the maximum principle, $\nabla F(x_0) = 0$ and $\Delta F(x_0) \leqslant 0$. Hence $LF(x_0) = \Delta F(x_0) - \nabla\phi(x_0)\nabla F(x_0) \leqslant 0$. These yield that at x_0 we have

$$\frac{\nabla\rho^2}{a^2-\rho^2} = \frac{\nabla\psi}{\psi}, \tag{2.17}$$

$$-L\rho^2\psi + \big(a^2-\rho^2\big)L\psi - 2\nabla\rho^2\cdot\nabla\psi \leqslant 0. \tag{2.18}$$

Hence

$$\frac{L\psi}{\psi} - \frac{L\rho^2}{a^2-\rho^2} - \frac{2|\nabla\rho^2|^2}{(a^2-\rho^2)^2} \leqslant 0. \tag{2.19}$$

By the generalized Laplacian comparison theorem, and using $|\nabla\rho^2| = 2\rho|\nabla\rho| = 2\rho$, we have:

$$L\rho^2 \leqslant 2 + 2(m-1)\bigl(1+\sqrt{K}\rho\bigr) \leqslant C\bigl(1+\sqrt{K}\bigr)\,\rho.$$

Combining this and (5.30) with (2.19), we have:

$$\begin{aligned}
0 &\geqslant \frac{L\psi}{\psi} - \frac{C(1+\sqrt{K}\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2}\\
&\geqslant -(m-1)K\psi - \left(2-\frac{2}{m-1}\right)\frac{\nabla\psi\cdot\nabla u}{u} + \frac{\psi^3}{m-1} - \frac{C(1+\sqrt{K}\rho)}{a^2-\rho^2} - \frac{8\rho^2}{(a^2-\rho^2)^2}.
\end{aligned}$$

Using (2.17) we have

$$\frac{\nabla\psi\cdot\nabla u}{\psi u} = \frac{2\rho\nabla\rho\cdot\nabla u}{(a^2-\rho^2)u} \leqslant \frac{2\psi\rho}{a^2-\rho^2}.$$

Replacing $\psi = F/(a^2-\rho^2)$ and this into the previous inequality we obtain

$$\begin{aligned}
0 &\geqslant \frac{1}{m-1}F^2 - \frac{4(m-2)}{m-1}\rho F - C\bigl(1+\sqrt{K}\rho\bigr)\bigl(a^2-\rho^2\bigr) - 8\rho^2 - (m-1)K\bigl(a^2-\rho^2\bigr)^2\\
&\geqslant \frac{1}{m-1}F^2 - \frac{4(m-2)}{m-1}aF - C\bigl(1+\sqrt{K}a\bigr)a^2 - 8a^2 - (m-1)Ka^4\\
&\geqslant \frac{1}{m-1}F^2 - 2C_1aF - C_2\bigl(1+\sqrt{K}a\bigr)^2a^2,
\end{aligned}$$

where C_1, C_2 are two constants depending only on m. This yields that

$$\begin{aligned}
F(x_0) &\leqslant \frac{m-1}{2}\Bigl(2C_1a + \sqrt{4C_1^2a^2 + 4(m-1)^{-1}\bigl(1+\sqrt{K}a\bigr)^2a^2}\Bigr)\\
&\leqslant (m-1)a\bigl(C_1(m) + C_2(m)\sqrt{K}a\bigr).
\end{aligned}$$

Therefore

$$\sup_{y\in B(x,a)}\left[\bigl(a^2-d^2(x,y)\bigr)\frac{|\nabla u(y)|}{u(y)}\right] \leqslant C_m a\bigl(1+\sqrt{K}a\bigr).$$

Restricting on $B(x,a/2)$, we get:

$$\frac{3a^2}{4}\sup_{y\in B(x,a/2)}\frac{|\nabla u(y)|}{u(y)} \leqslant C_m a\bigl(1+\sqrt{K}a\bigr).$$

Hence

$$\sup_{y\in B(x,a/2)} \frac{|\nabla u(y)|}{u(y)} \leqslant C_m\left(\frac{1}{a}+\sqrt{K}\right). \qquad \square$$

Using the same argument as in Schoen and Yau [63], we can derive the following results from the generalized Schoen–Yau inequality. To save the length of the paper, we omit the proofs.

Corollary 2.4. *Suppose that $Ric_{m,n}(L) \geqslant -(m-1)K$, where $K \geqslant 0$ is a constant. Let u be a solution of $Lu=0$ in the geodesic ball $B(x,a)$, $\forall x \in M$. Then*

$$\sup_{B(x,a/2)} |\nabla u| \leqslant C_m\left(\frac{1+\sqrt{K}a}{a}\right) \sup_{B(x,a)} |u|,$$

where C_m is a constant depending only on m.

Corollary 2.5 (Harnack inequality). *Suppose that $Ric_{m,n}(L) \geqslant -(m-1)K$, where $K \geqslant 0$ is a constant. Let u be a positive solution of $Lu=0$ in the geodesic ball $B(x,a)$, $\forall x \in M$. Then*

$$\sup_{B(x,a/2)} u \leqslant \mathrm{e}^{2aC_m(\frac{1+\sqrt{K}a}{a})} \inf_{B(x,a/2)} u,$$

where C_m is a constant depending only on m.

Similarly, we have the following result for the solution of $(L-\lambda)u=0$ for $\lambda>0$.

Theorem 2.6. *Suppose that $Ric_{m,n}(L) \geqslant -(m-1)K$, where $K \geqslant 0$ is a constant. Let u be a positive solution of $(L-\lambda)u=0$, where $\lambda>0$ is a constant. Then there exists a constant $C(m,K,\lambda)$ depending only on m, K and λ such that*

$$|\nabla u| \leqslant C(m,K,\lambda)u.$$

3. Conservativeness of heat semigroup

In [22], E.B. Davies extended Karp–Li's conservative criterion to symmetric diffusion operators. He proved that if there exist two constants C_1 and C_2 such that $\mu(B(x,R)) \leqslant C_1\mathrm{e}^{C_2R^2}$ for some (and hence for all) $x \in M$ then the heat semigroup $P_t=\mathrm{e}^{tL}$ is conservative. This can be considered as a special case of the generalized Grigor'yan criterion for the conservativeness of Dirichlet form, see Sturm [66], which says that if for some $x \in M$,

$$\int_1^\infty \frac{r\,\mathrm{d}r}{\log\mu(B(x,r))} = +\infty,$$

then $P_t = \mathrm{e}^{tL}$ is conservative. In view of these, Theorem 1.4 follows from the following

Lemma 3.1. *Under the same condition of Theorem* 1.4*, we have*

$$\mu\big(B(o,R)\big) \leqslant \mu\big(B(o,r)\big)\left(\frac{R}{r}\right)^m \exp\big((m-1)\sqrt{K(R)}\,(R-r)\big), \quad \forall R > r > 0.$$

Proof. Let $\rho(x) = d(x,o)$. Then the diffusion property of L implies:

$$L\rho^2 = 2\rho L\rho + 2|\nabla\rho|^2 \leqslant 2 + 2\rho L\rho, \quad \text{on } M \setminus cut(o).$$

By Theorem 1.1, we have $L\rho \leqslant a(\rho)$, where $a(\rho)$ is the solution of the Riccati equation

$$-a'(\rho) = -K(\rho) + \frac{a^2(\rho)}{m-1}, \quad \text{with } \lim_{x\to 0} xa(x) = m-1.$$

Let $a_{K(R)}$ be the solution of the Riccati equation

$$-a'(\rho) = -K(R) + \frac{a^2(\rho)}{m-1}, \quad \text{with } \lim_{x\to 0} xa(x) = m-1.$$

The Strum–Liouville comparison theorem of Riccati equation implies

$$a\big(\rho(x)\big) \leqslant a_{K(R)}\big(\rho(x)\big) = (m-1)\sqrt{K(R)}\,\coth\big[\sqrt{K(R)}\,\rho(x)\big], \quad \forall x \in B(o,R) \setminus cut(o).$$

Combining this together with $\sqrt{K(R)}\,\rho\coth(\sqrt{K(R)}\,\rho) \leqslant 1 + \sqrt{K(R)}\rho$, we have

$$\begin{aligned} L\rho^2 &\leqslant 2 + 2(m-1)\sqrt{K(R)}\,\rho\coth\big[\sqrt{K(R)}\rho(x)\big] \\ &\leqslant 2m + 2(m-1)\sqrt{K(R)}\,\rho, \quad \forall x \in B(o,R) \setminus cut(o). \end{aligned}$$

Integrating over $B(o,R)$ and since $\mu(cut(o)) = 0$, we have:

$$\begin{aligned} \int_{B(o,R)} L\rho^2(x)\,\mathrm{d}\mu(x) &\leqslant 2m\mu\big(B(o,R)\big) + 2(m-1)\sqrt{K(R)} \int_{B(o,R)} \rho(x)\,\mathrm{d}\mu(x) \\ &\leqslant 2\big[m + (m-1)\sqrt{K(R)}R\big]\mu\big(B(o,R)\big). \end{aligned}$$

On the other hand, applying Green's formula, we have:

$$\int_{B(o,R)} L\rho^2(x)\,\mathrm{d}\mu(x) = \int_{\partial B(o,R)} \frac{\partial\rho^2}{\partial\nu}(x)\,\mathrm{d}\mu_\sigma(x) = 2R\mu_\sigma\big(\partial B(o,R)\big),$$

where $\frac{\partial}{\partial\nu}$ denotes the exterior normal derivative, μ_σ denotes the weight area-measure induced by the weight volume measure μ. Moreover, using the co-area formula, we have

$$\mu_\sigma\big(\partial B(o,R)\big)=\frac{\partial\mu(B(o,r)}{\partial r}\bigg|_{r=R}.$$

Hence

$$\int_{B(o,R)} L\rho^2(x)\,\mathrm{d}\mu(x)=2R\frac{\mathrm{d}\mu(B(o,R)}{\mathrm{d}R}.$$

Let $V(R)=\mu(B(o,R))$. Then

$$R\frac{\mathrm{d}V_R}{\mathrm{d}R}\leqslant\big(m+(m-1)\sqrt{K(R)}\,R\big)V_R.$$

Thus

$$\frac{\mathrm{d}\log V_R}{\mathrm{d}R}\leqslant\frac{m}{R}+(m-1)\sqrt{K(R)}.$$

Integrating with respect R from r to R, we have

$$\begin{aligned}V_R&\leqslant V_r\left(\frac{R}{r}\right)^m\exp\left((m-1)\int_r^R\sqrt{K(s)}\,\mathrm{d}s\right)\\&\leqslant V_r\left(\frac{R}{r}\right)^m\exp\big((m-1)\sqrt{K(R)}\,(R-r)\big).\end{aligned}$$

The proof of Lemma 3.1 is completed. □

Remark 3.1. The above proof of Theorem 1.4 is purely geometric. Using the Laplace comparison theorem in Riemannian geometry, the comparison theorem of one-dimensional stochastic differential equations and the Itô–Skorokhod formula for the radial part of Brownian motion, Hsu [35,36] proved Theorem 1.4 in the case where $\phi\equiv 0$. This probabilistic method has been further developed by Qian [57] to establish his conservativeness criterion for L-diffusion process mentioned in Remark 1.3. In the case of symmetric diffusion operator $L=\Delta-\nabla\phi\cdot\nabla$, we can use this approach to give a probabilistic proof of Theorem 1.4. To save the length of the paper, we omit it. See Section 4 for Kendall's Itô–Skorokhod formula for the radial part of the L-diffusion process.

Remark 3.2. By the generalized Laplacian comparison theorem, if $Ric_{m,n}(L)\geqslant -(m-1)K^2$, where $K\geqslant 0$ is a constant, we can see from the proof of Theorem 1.4 that

$$L\rho(\cdot\,,x)|_y \leqslant \frac{m-1}{\rho(y,x)}\big(1+K\rho(y,x)\big), \quad \forall y \in M \setminus cut(x). \tag{3.20}$$

In the case where $L = \Delta$ and $m = n$, the above inequality is well-known in geometry. Moreover, see Appendix in Yau [73] and Remark 1.5 in Schoen [62], it can be shown that a corresponding global inequality holds in the sense of distribution: if $Ric \geqslant -(n-1)K^2$, then for all $\psi \in C_0^\infty(M, \mathbb{R}^+)$,

$$\int_M \rho(y,x)\Delta\psi(y)\,\mathrm{d}y \leqslant (n-1)\int_M \frac{1+K\rho(y,x)}{\rho(y,x)}\psi(y)\,\mathrm{d}y. \tag{3.21}$$

For a proof, see Appendix (iv) in Yau [73], Proposition 1.1.1 in Schoen and Yau [63], or Section 1.6 in Schoen [62]. In the general case, $Ric_{m,n}(L) \geqslant -(m-1)K^2$ also implies the following global inequality in the sense of distribution: for all $\psi \in C_0^\infty(M, \mathbb{R}^+)$,

$$\begin{aligned}&\int_M \rho(y,x)L\psi(y)\,\mathrm{d}\mu(y) \leqslant (m-1)\int_M \frac{1+K\rho(y,x)}{\rho(y,x)}\psi(y)\,\mathrm{d}\mu(y),\\ &\quad \forall \psi \in C_0^\infty(M),\ \psi \geqslant 0.\end{aligned} \tag{3.22}$$

In fact, since $\phi \in C^2(M)$, the measure μ is equivalent to the volume measure dx. As the Hausdorff measure of $cut(x)$ is zero, we see that $\mu(cut(x)) = 0$ for all $x \in M$. Hence, if we denote $\Omega = M \setminus cut(x)$, then

$$\int \rho(y,x)L\psi(y)\,\mathrm{d}\mu(y) = \int_\Omega \rho(y,x)L\psi(y)\,\mathrm{d}\mu(y).$$

Defining $\Omega_\varepsilon = \{y \in \Omega,\ d(y, cut(x)) \geqslant \varepsilon\}$, then Ω_ε tends to Ω as ε tends to zero. Applying the Stokes formula and the Green formula on Ω_ε and using (3.20), we can prove that

$$\begin{aligned}\int_M \rho(y,x)L\psi(y)\,\mathrm{d}\mu(y) \leqslant{}& (m-1)\int_M \frac{1+K\rho(y,x)}{\rho(y,x)}\psi(y)\,\mathrm{d}\mu(y)\\ &- \lim_{\varepsilon\to 0}\int_{\partial\Omega_\varepsilon} \psi(y)\frac{\partial\rho(y,x)}{\partial\nu}\,\mathrm{d}\mu_\sigma(y).\end{aligned}$$

As Ω is star-sharped, so is Ω_ε. Hence, $\frac{\partial\rho(y,x)}{\partial\nu} \geqslant 0$. Therefore the second term in the last inequality is non-negative. This proves (3.22). To save the length of the paper, we omit the detail of the proof.

By the well-known equivalence between the conservativeness and the L^∞-uniqueness of the Cauchy problem of heat equation, see [24,30,33,36], we obtain immediately:

Theorem 3.2. *Under the same condition as in Theorem* 1.4*, for any bounded continuous function $f \in L^\infty(M)$, there exists a unique bounded solution to the heat equation $\partial_t u = Lu$ with the initial condition $u(0,x) = f(x)$ for all $x \in M$.*

4. C_0-diffusion property of heat semigroup

To prove Theorem 1.5, we follow the method used in Azencott [5] and developed in Hsu [35,36], see also Qian [57]. By Theorem 1.4, the L-diffusion process is conservative. Let $\{X_t,\ t < \infty\}$ be the L-diffusion process starting at $x \in M$. By Azencott [5], we need only to prove that for any closed geodesic ball $K = B(o,R)$, where R is any fixed constant, we have

$$\lim_{d(x,o)\to\infty} P_x(T_K < t) = 0, \tag{4.23}$$

where $T_K := \inf\{t > 0\colon\ X_t \in K\}$ is the entrance time of $\{X_t,\ t < \infty\}$ in $K = B(o,R)$. Let $\sigma_0 = 0$, and for all $n \in \mathbb{N}$,

$$\begin{aligned}\tau_n &= \inf\bigl\{t > \sigma_n\colon\ d(X_t, X_{\sigma_n}) = 1\bigr\}, \quad n \geqslant 0,\\ \sigma_n &= \inf\bigl\{t \geqslant \tau_{n-1}\colon\ \rho(X_t) = \rho(x) - n\bigr\}, \quad n \geqslant 1.\end{aligned}$$

That is, σ_n is the entrance time of the L-diffusion process X_t in the geodesic ball $B(o, \rho(x) - n)$, $\theta_n := \tau_n - \sigma_n$ is the amount of time during which the L-diffusion process moves from $X_{\sigma_n} \in \partial B(o, \rho(x) - n)$ to $X_{\tau_n} \in \partial B(X_{\sigma_n}, 1)$, and $\sigma_{n+1} - \tau_n$ is the amount of time during which the L-diffusion process leaves from $\partial B(X_{\sigma_n}, 1)$ and hits $\partial B(o, \rho(x) - (n+1))$. Let

$$\theta_n = \tau_n - \sigma_n.$$

Then

$$T_K \geqslant \sigma_{[\rho(x)-R]} \geqslant \theta_0 + \theta_1 + \cdots + \theta_{[\rho(x)-R-1]},$$

where $[\rho(x) - R]$ denotes the integer part of $\rho(x) - R$. The key point is to prove that there exist two constants $C_1 > 0$ and $C_2 > 0$ such that for all $n \geqslant 0$,

$$P_x\Bigl(\theta_k \leqslant C_1\big/\sqrt{K\bigl(\rho(x) - n + 1\bigr)}\Bigr) \leqslant \mathrm{e}^{-C_2 K(\rho(x)-n+1)}. \tag{4.24}$$

To this end, we use Kendall's Itô–Skorokhod formula. In fact, see Kendall [38], under the probability measure P_x, there exists a standard one-dimensional Brownian motion β_t such that $r(X_t) = d(X_t, x)$ can be decomposed into

$$r(X_t) = \beta_t + \frac{1}{2}\int_0^t Lr(X_s)\,\mathrm{d}s - L_t, \tag{4.25}$$

where L_t is a nondecreasing process which is increasing only on $\{t\colon\ X_t \in cut(x)\}$. For a proof, see also Remark 4.1. Moreover, using the Kendall decomposition and the Girsanov transformation, we have (cf. Qian [57]),

$$d(X_t, X_{\sigma_n}) = \beta_t - \beta_{\sigma_n} + \frac{1}{2}\int_{\sigma_n}^{t} Ld(X_s, X_{\sigma_n})\,\mathrm{d}s - (L_t - L_{\sigma_n}).$$

Note that

$$d^2(X_t, X_{\sigma_n}) = 2\int_0^t d(X_s, X_{\sigma_n}) d\big(d(X_s, X_{\sigma_n})\big) + \big\langle d(X_\cdot, X_{\sigma_n}), d(X_\cdot, X_{\sigma_n})\big\rangle_t.$$

Since $\langle d(X_\cdot, X_{\sigma_n}), d(X_\cdot, X_{\sigma_n})\rangle_t = \langle \beta, \beta\rangle_t = t$ and $L_t - L_{\sigma_n}$ is a nondecreasing positive process on $[\sigma_n, \tau_n]$, we have

$$d^2(X_t, X_{\sigma_n}) \leqslant 2\int_{\sigma_n}^{t} d(X_s, X_{\sigma_n})\,\mathrm{d}\beta_s + \int_{\sigma_n}^{t} d(X_s, X_{\sigma_n}) Ld(X_s, X_{\sigma_n})\,\mathrm{d}s + t - \sigma_n. \tag{4.26}$$

For all $t \in [\sigma_n, \tau_n]$, $X_t \in B(X_{\sigma_n}, 1) \subset B(o, \rho(x) - n + 1)$. While

$$Ric_{m,n}(L)(y) \geqslant -K\big(\rho(x) - n + 1\big), \quad \forall y \in B\big(o, \rho(x) - n + 1\big).$$

By the generalized Laplacian comparison theorem, see Corollary 1.2, we have

$$\begin{aligned} &Ld(y, X_{\sigma_n}) \leqslant (m-1)\sqrt{K(\rho - n + 1)}\coth\big(\sqrt{K(\rho - n + 1)}\, d(y, X_{\sigma_n})\big)\\ &\quad \text{on } B\big(o, \rho(x) - n + 1\big) \setminus cut(X_{\sigma_n}). \end{aligned}$$

Using $a\coth a \leqslant 1 + a$ for all $a \geqslant 0$, we obtain

$$d(X_t, X_{\sigma_n}) Ld(X_t, X_{\sigma_n}) \leqslant (m-1)\big(1 + d(X_t, X_{\sigma_n})\sqrt{K(\rho - n + 1)}\,\big).$$

Taking $t = \tau_n$ in (4.26) and since $d(X_s, X_{\sigma_n}) \leqslant d(X_{\tau_n}, X_{\sigma_n}) = 1$ for all $s \in [\sigma_n, \tau_n]$, we obtain

$$1 \leqslant 2\int_{\sigma_n}^{\tau_n} d(X_s, X_{\sigma_n})\,\mathrm{d}\beta_s + \Big[(m-1)\Big(1 + \sqrt{K\big(\rho(x) - n + 1\big)}\Big) + 1\Big](\tau_n - \sigma_n).$$

Without loss of the generality we may assume $K(\rho(x) - n + 1) \geqslant 1$ then

$$1 \leqslant 2\int_{\sigma_n}^{\tau_n} d(X_s, X_{\sigma_n})\,\mathrm{d}\beta_s + 2m\sqrt{K\big(\rho(x) - n + 1\big)}\,(\tau_n - \sigma_n).$$

This yields that, for any enough small constant $C_1 > 0$,

$$P_x\big(\tau_n - \sigma_n \leqslant C_1/\sqrt{K\big(\rho(x) - n + 1\big)}\big) \leqslant P_x\left(\int_{\sigma_n}^{\tau_n} d(X_s, X_{\sigma_n})\, \mathrm{d}\beta_s \geqslant \frac{1}{8}\right).$$

Based on Lévy's criterion and the random time change, the standard method as used in [34, 36] proves that

$$P_x\left(\int_{\sigma_n}^{\tau_n} r(X_s)\, \mathrm{d}\beta_s \geqslant \frac{1}{8}\right) \leqslant \exp\big(-C_2\sqrt{K\big(\rho(x) - n + 1\big)}\big).$$

Therefore we have proved (4.24).

Then we can follow the same argument used in Hsu [36] to obtain

$$P_x(T_K \leqslant t) \leqslant \sum_{n=1}^{N(x,t)} \mathrm{e}^{-C_2\sqrt{K(\rho(x)-n+1)}},$$

where $N(x,t)$ is the smallest integer such that

$$\sum_{n=1}^{N} \frac{1}{\sqrt{K(\rho(x) - n + 1)}} > \frac{t}{C_1}.$$

As $\int_1^\infty K(s)^{-1/2}\, \mathrm{d}s = \infty$, such $N(x,t)$ exists for sufficiently large $\rho(x)$ and moreover $N(x,t) \leqslant [\rho(x) - R]$. Then we can obtain

$$P_x(T_K \leqslant t) \leqslant \int_{\rho(x)-N(x,t)+1}^{\rho(x)+1} \mathrm{e}^{-C_2\sqrt{K(r)}}\, \mathrm{d}r.$$

By Hsu [35,36], the latter term tends to zero as $\rho(x) \to \infty$. This proves the desired key estimate (4.23) for $K = B(o, R)$.

Remark 4.1 (*Proof of* (4.1)). In the Appendix (see Formula (6)) of Yau [73], it was proved that for any smooth non-negative function ψ with compact support on M, we have

$$\int_M \rho(x)\Delta\psi(x)\, \mathrm{d}x \leqslant \int_{M\setminus cut(x_0)} \psi(x)\Delta\rho(x)\, \mathrm{d}x. \tag{4.27}$$

Here x_0 is a fixed point in M, $\rho(x) = d(x_0, x)$. If we let $\Delta\rho$ to denote the Laplacian of ρ in the sense of distribution, then, as pointed out in Hsu [35], the above Yau's distributional inequality (4.27) implies that the distribution

$$\nu_\Delta := \Delta\rho I_{M\setminus cut(x_0)} - \Delta\rho$$

is non-negative on non-negative test functions. Similarly to Yau's distributional inequality (4.27) for the Laplace–Beltrami operator, for all test functions $\psi \in C_0^\infty(M, \mathbb{R}^+)$, we have:

$$\int_M \rho(x) L\psi(x)\,\mathrm{d}\mu(x) \leqslant \int_{M\setminus cut(x_0)} \psi(x) L\rho(x)\,\mathrm{d}\mu(x). \tag{4.28}$$

Indeed, since μ is equivalent to dx and since the Hausdorff measure of $cut(x_0)$ is zero, we have $\mu(\mathrm{supp}(\psi) \cap cut(x_0)) = 0$. Hence

$$\int_M \rho(x) L\psi(x)\,\mathrm{d}\mu(x) = \int_{M\setminus cut(x_0)} \rho(x) L\psi(x)\,\mathrm{d}\mu(x).$$

Now take a sequence of star-sharp approximation $\Omega_\varepsilon = \{y \in M \setminus cut(x_0)\colon\ d(y, cut(x_0)) \geqslant \varepsilon\}$. Applying the Stokes formula and the Green formula on Ω_ε, we have:

$$\begin{aligned}\int_M \rho(x) L\psi(x)\,\mathrm{d}\mu(x) &= \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} L\psi(x)\rho(x)\,\mathrm{d}\mu(x) - \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \psi(x) \frac{\partial\rho(x)}{\partial\nu}\,\mathrm{d}\mu_\sigma(x) \\ &= \int_{M\setminus cut(x_0)} L\psi(x)\rho(x)\,\mathrm{d}\mu(x) - \lim_{\varepsilon\to 0} \int_{\partial\Omega_\varepsilon} \psi(x) \frac{\partial\rho(x)}{\partial\nu}\,\mathrm{d}\mu_\sigma(x).\end{aligned}$$

Here $\frac{\partial}{\partial\nu}$ denotes the exterior normal derivative. Since Ω_ε is star-sharp, we have $\frac{\partial\rho(x)}{\partial\nu} \geqslant 0$. Hence $\int_{\partial\Omega_\varepsilon} \psi(x) \frac{\partial\rho(x)}{\partial\nu}\,\mathrm{d}\mu_\sigma(x)$ is non-negative. The inequality (4.28) follows. Equivalently to say, the distribution

$$\nu_L := L\rho I_{M\setminus cut(x_0)} - L\rho$$

is non-negative on non-negative test functions. Similarly to the case discussed in Hsu [35] for $L = \Delta$, by the Riesz representation theorem, ν_L is a Radon measure on M supported on $cut(x_0)$. Hence, the distribution $L\rho$ is indeed a Radon measure on M. Using the generalized Itô formula, one proves immediately the Kendall decomposition (4.25) in which L_t is the continuous positive additive functional associated with the measure ν_L.

5. Harnack inequalities and heat kernel estimates

In order to prove the L^1-Liouville theorem for the solution of $Lu = 0$ and the L^1-uniqueness of the heat semigroup $P_t = \mathrm{e}^{tL}$, we need to establish Li–Yau parabolic Harnack inequalities and to prove some heat kernel estimates.

5.1. Upper bound estimate via Li–Yau Harnack inequalities

In this subsection we prove the Li–Yau differential Harnack inequality and the Li–Yau Harnack inequality for the heat semigroup generated by $L = \Delta - \nabla\phi \cdot \nabla$ under the curvature-dimension condition $CD(K_{m,n}, m)$, where $K_{m,n}(x)$ is a radial function depended on $\rho(x) = d(x, o)$. These estimates allow us to deduce the Li–Yau upper bound estimate of the heat kernel on geodesic balls. In principle we follow the method of Li and Yau [43] except that we are dealing with the heat semigroup of the diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ rather than the heat semigroup of the Laplace–Beltrami operator Δ. To the reader who is very familiar in Li–Yau's technique, it might be thought that one does not need to reproduce the proofs of the Li–Yau Harnack inequalities for the heat semigroup of L as it is well known that the Bakry–Emery Ricci curvature of L plays a very similar role as the Ricci curvature for Δ. However we still reproduce the proof of the differential Harnack inequality in detail since our aim here is to establish these inequalities under a natural geometric and analytic condition which we hope to be as sharp as possible.

Theorem 5.1. *Let M be a complete Riemannian manifold, $L = \Delta - \nabla\phi \cdot \nabla$, $\phi \in C^2(M)$. Let $-K(2R)$ be a lower bound of the Bakry–Emery Ricci curvature*

$$Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}$$

on the geodesic ball $B(o, 2R)$. Let $u(x,t)$ be a positive solution of the heat equation

$$\partial_t u = Lu$$

on $M \times (0, T]$. Then, for any $\alpha > 1$, we have

$$\frac{|\nabla u|^2}{u^2} - \frac{\alpha u_t}{u} \leqslant \frac{C\alpha^2}{R^2}\left(1 + R\sqrt{K(2R)} + \frac{\alpha^2}{\alpha - 1}\right) + \frac{m\alpha^2}{2t} + \frac{m\alpha^2 K(2R)}{(\alpha-1)\sqrt{2}}. \tag{5.29}$$

Proof. The proof is similar to the one of Theorem 2.1 in [43]. Let $f = \log u$. The diffusion property $Lg(u) = g'(u)Lu + g''(u)|\nabla u|^2$ for all $g \in C^2(\mathbb{R}, \mathbb{R})$ implies that

$$\partial_t f = Lf + |\nabla f|^2.$$

Define

$$F(x,t) = t\big(|\nabla f|^2 - \alpha f_t\big).$$

Let η be a C^2-function on $[0, \infty)$ such that

$$\eta(r) = \begin{cases} 1 & \text{on } [0,1], \\ 0 & \text{on } [2, \infty), \end{cases}$$

with $-C_1\eta^{1/2}(r) \leqslant \eta'(r) \leqslant 0$ and $\eta''(r) \geqslant -C_2$, where C_1, C_2 are two positive constants. Let $\rho(x) = d(o, x)$. Define

$$\psi(x) = \eta\big(\rho(x)/R\big).$$

Since $\rho(x)$ is Lipschitz on the cut locus of o, ψF is a Lipschitz function with support in $B(o, 2R) \times [0, \infty)$. As explained in Li and Yau [43], an argument of Calabi allows us to apply the maximum principle to ψF. Let $(x_0, t_0) \in M \times [0, T]$ be a point where ψF achieves its maximum. We assume $\psi F(x_0, t_0) > 0$, otherwise the theorem holds trivially. At (x_0, t_0) we have

$$\nabla(\psi F) = 0, \qquad \psi F_t = \frac{\partial}{\partial t}(\psi F) \geqslant 0, \qquad \Delta(\psi F) \leqslant 0.$$

By the generalized Laplacian comparison theorem, as $Ric_{m,n}(L) \geqslant -(m-1)K(2R)$, we have

$$L\rho \leqslant (m-1)\sqrt{K(2R)}\coth\big(\sqrt{K(2R)}\,\rho\big).$$

Note that

$$L\psi(x) = \frac{\eta'(\rho(x)/R)Lr}{R} + \frac{\eta''(\rho(x)/R)|\nabla\rho(x)|^2}{R^2}.$$

By the conditions on η, we have

$$L\psi(x) \geqslant -\frac{C_1}{R}(m-1)\sqrt{K(2R)}\coth\big(\sqrt{K(2R)}\,R\big) - \frac{C_2}{R^2}. \tag{5.30}$$

Under the curvature-dimension $CD(K(2R), m)$ condition, for any $\alpha > 1$, we have

$$L|\nabla f|^2 \geqslant \frac{2}{m}|Lf|^2 + 2\nabla(Lf)\cdot\nabla f - 2K|\nabla f|^2.$$

Hence

$$\begin{aligned}(L-\partial_t)(tf_t) &= t\partial_t Lf - f_t - tf_{tt} = tLf_t - f_t - t\partial_t\big(Lf + |\nabla f|^2\big)\\ &= tLf_t - f_t - t(Lf_t + 2\nabla f\cdot\nabla f_t) = -2t\nabla f\cdot\nabla f_t - f_t.\end{aligned}$$

These yield

$$\begin{aligned}(L-\partial_t)F &= (L-\partial_t)\big(t|\nabla f|^2 - \alpha t f_t\big)\\ &\geqslant \frac{2t}{m}|Lf|^2 + 2t\nabla(Lf)\cdot\nabla f - 2tK|\nabla f|^2 - |\nabla f|^2 - 2t\nabla f\cdot\nabla f_t\\ &\quad + 2\alpha t\nabla f\cdot\nabla f_t + \alpha f_t\end{aligned}$$

$$\geqslant \frac{2t}{m}\big(|\nabla f|^2 - f_t\big)^2 - 2t\nabla f\cdot\nabla\big(|\nabla f|^2 - f_t\big) - 2Kt|\nabla f|^2$$
$$- \big(|\nabla f|^2 - \alpha f_t\big) + 2(\alpha-1)t\nabla f\cdot\nabla f_t.$$

Therefore

$$(L-\partial_t)F \geqslant \frac{2t}{m}\big(|\nabla f|^2 - f_t\big)^2 - 2Kt|\nabla f|^2 - t^{-1}F - 2\langle\nabla f, \nabla F\rangle. \tag{5.31}$$

Replacing (5.30) and (5.31) into

$$L(\psi F) = (L\psi)F + 2\langle\nabla\psi, \nabla F\rangle + \psi LF,$$

we have:

$$\begin{aligned} L(\psi F) \geqslant{}& -F\big(C_1R^{-1}(m-1)\sqrt{K(2R)}\coth\big(\sqrt{K(2R)}\,R\big) + C_2R^{-2}\big)\\ &+ 2\big\langle\nabla\psi, \nabla(\psi F)\big\rangle\psi^{-1} - 2F|\nabla\psi|^2\psi^{-1}\\ &+ \psi\left[F_t - 2\langle\nabla f, \nabla F\rangle + \frac{2}{m}t\big(|\nabla f|^2 - f_t\big)^2 - t^{-1}F - 2K(2R)t|\nabla f|^2\right]. \end{aligned}$$

Note that at (x_0, t_0) we have

$$L(\psi F) = \Delta(\psi F) - \nabla\phi\cdot\nabla(\psi F) \leqslant 0.$$

Hence

$$\begin{aligned} 0 \geqslant{}& -F\big(C_1R^{-1}(m-1)\sqrt{K(2R)}\coth\big(\sqrt{K(2R)}\,R\big) + C_2R^{-2}\big) - 2F|\nabla\psi|^2\psi^{-1}\\ &+ 2F\langle\nabla f, \nabla\psi\rangle + \frac{2}{m}t_0\psi\big(|\nabla f|^2 - f_t\big)^2 - t_0^{-1}\psi F - 2K(2R)t_0\psi|\nabla f|^2. \end{aligned}$$

Using the same argument as used in Li and Yau [43], see pp. 161–162, or taking $\gamma = \theta = \varepsilon = 0$ there, we can show that at the maximum point $(x_0, t_0) \in M \times [0, T]$, we have

$$\begin{aligned} \psi F \leqslant (\mathrm{RHS}) :={}& C_3\alpha^2t_0R^{-2}\big(1 + R\sqrt{K(2R)} + \alpha^2(\alpha-1)^{-1}\big)\\ &+ \frac{m\alpha^2}{2} + \frac{t_0m\alpha^2(\alpha-1)^{-1}K(2R)}{\sqrt{2}}. \end{aligned}$$

This yields, for all $(x, T) \in M \times \{T\}$, $(\psi F)(x, T) \leqslant (\mathrm{RHS})$. In particular, for all $x \in B(o, R)$,

$$\begin{aligned} T\big(|\nabla f|^2 - \alpha f_t\big)\big|_{t=T} \leqslant{}& C_3\alpha^2t_0R^{-2}\big(1 + R\sqrt{K(2R)} + \alpha^2(\alpha-1)^{-1}\big)\\ &+ \frac{m\alpha^2}{2} + \frac{t_0m\alpha^2(\alpha-1)^{-1}K(2R)}{\sqrt{2}}. \end{aligned}$$

Since $t_0 \leqslant T$, this implies the desired inequality by taking $t = T$ in (5.29). $\square$

The differential Harnack inequality implies the following parabolic Harnack inequality.

Theorem 5.2. *Let M be a complete Riemannian manifold, $L = \Delta - \nabla\phi \cdot \nabla$, $\phi \in C^2(M)$. Let $-K(2R)$ be a lower bound of the Bakry–Emery Ricci curvature*

$$Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}$$

on the geodesic ball $B(o, 2R)$. Let $u(x,t)$ be a positive solution of the heat equation

$$\partial_t u = Lu$$

on $M \times (0, T]$. Then, for any $\alpha > 1$, $0 < t_1 < t_2 < T$ and $x, y \in B(o, R)$, we have

$$u(x, t_1) \leqslant u(y, t_2)\left(\frac{t_2}{t_1}\right)^{m\alpha/2} \exp\left(A(t_2 - t_2) + \frac{\alpha d^2(x,y)}{4(t_2 - t_2)}\right), \tag{5.32}$$

where

$$A = C\left[\alpha R^{-1}\sqrt{K(2R)} + \alpha^3(\alpha-1)^{-1}R^{-2} + \alpha(\alpha-1)^{-1}K(2R)\right].$$

Proof. By integrating $\mathrm{d}/\mathrm{d}s\,(\log u)$ along the curve $\eta(s) = (\gamma(s), (1-s)t_2 + st_1)$, where γ is a geodesic between x and y, and using the same argument as used in the proof of Theorem 2.1 in Li and Yau [43], Theorem 5.2 follows from Theorem 5.1. $\square$

Corollary 5.3. *Under the same condition and the same notation as in Theorem* 5.2, *we have the mean value inequality*

$$u(x, t_1) \leqslant \left(\int_{B(x,R)} u^p(y, t_2)\,\mathrm{d}\mu_R(y)\right)^{1/p} \cdot \left(\frac{t_2}{t_1}\right)^{m\alpha/2} \cdot \exp\left(\frac{\alpha R^2}{4(t_2 - t_1)} + A(t_2 - t_1)\right),$$

where $p > 0$, $\alpha > 1$, $0 < t_1 < t_2$, $\mathrm{d}\mu_R(y) = \mu^{-1}(B(o,R))\,\mathrm{d}\mu(y)$.

Proof. This follows from Theorem 5.2 by integrating (5.32) over $B(o, R)$. $\square$

Finally, using the same argument as used in the proof of Theorem 3.3 in Li and Yau [43], the parabolic Harnack inequality implies the following upper bound estimate of the heat kernel $H(x, y, t)$ for the diffusion operator L. To save the length of the paper we leave the details of this proof to the reader.

Theorem 5.4. *Let M be a complete Riemannian manifold, $L = \Delta - \nabla\phi \cdot \nabla$, $\phi \in C^2(M)$. Let $-K(2R)$ be a lower bound of the Bakry–Emery Ricci curvature*

$$Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n}$$

on the geodesic ball $B(o, 2R)$. Then, for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} H(x, y, t) \leqslant C(\varepsilon)\mu^{-1/2}\big(B_x(\sqrt{t}\,)\big)\mu^{-1/2}\big(B_y(\sqrt{t}\,)\big) \\ \times \exp\left(\frac{-d^2(x, y)}{(4+\varepsilon)t} + \alpha\varepsilon\big(K(R) + R^{-2}\big)t\right) \end{aligned} \tag{5.33}$$

for all $x, y \in B_{x_0}(R) := B(x_0, R)$, $t > 0$ and some constant α depending only on n.

To end this section, let us mention that the Li–Yau differential Harnack inequality for the heat equation $\partial_t u = \Delta u$ has been improved by Bakry and Qian [12]. Using the same technique based on the Laplacian type comparison theorem and the curvature-dimension condition $CD(K, m)$, we can prove that the Bakry–Qian differential Harnack inequality holds for diffusion operator L with $CD(K, m)$ condition.

Theorem 5.5. *Suppose that $L = \Delta - \nabla\phi \cdot \nabla$ satisfies the curvature-dimension condition*

$$Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geqslant -K$$

with $K \geqslant 0$. Let $f = \log u$, where u is a solution to the heat equation $\partial_t u = Lu$. Then

$$|\nabla f|^2 - f_t \leqslant \sqrt{mK}\sqrt{|\nabla f|^2 + \frac{m}{2t} + \frac{mK}{4}} + \frac{m}{2t}.$$

5.2. L^2-estimate of heat kernel outside geodesic balls

In this subsection we prove the following exterior L^2-estimate for the heat kernel.

Proposition 5.6. *Let $K(2R)$ be the lower bound of $Ric_{m,n}(L)$ on a geodesic ball $B(o, 2R)$. Then, for all $x \in M$, $t > 0$ and $\alpha > 1$,*

$$I = \int\limits_{M \setminus B(o,R/2)} H^2(x, y, t)\,\mathrm{d}\mu(y) \leqslant C\mu^{-1}\big(B\big(x, \sqrt{t}\,\big)\big)\mathrm{e}^{\frac{3At}{2} - \frac{(R-2d(x,o))^2}{20t}}, \tag{5.34}$$

where

$$A = C\big[\alpha R^{-1}\sqrt{K(2R)} + \alpha^3(\alpha-1)^{-1}R^{-2} + \alpha(\alpha-1)^{-1}K(2R)\big].$$

To prove the above proposition, we need the following L^2-monotone integral inequality.

Lemma 5.7. *Let $u(t,x)$ be an $L^2(\mu)$-solution of the heat equation $\partial_t u = Lu$ with the initial data $u(0,\cdot) \in L^2(\mu)$. Assuming that $g(x,t) \in C^1(M \times [0,\infty))$ satisfies*

$$\frac{\partial g}{\partial t} + \frac{1}{4}|\nabla g|^2 = 0, \quad g \leqslant 0.$$

Then for any $R > 0$, $T > 0$ and any $x \in M$ we have

$$\int\limits_{B(x,R)} \mathrm{e}^{\frac{1}{2}g(y,T)} u^2(T,y)\,\mathrm{d}\mu(y) \leqslant \int\limits_{B(x,R)} \mathrm{e}^{\frac{1}{2}g(y,0)} u^2(0,y)\,\mathrm{d}\mu(y).$$

Proof. The proof modifies the argument used in the one of Lemma VI.2 in Schoen and Yau [63], see also the proof of Lemma 3.2 in Li and Yau [43]. It is based on the integration by parts formula for the weight volume measure μ, the standard method using cut-off function on geodesic balls together with the Cauchy–Schwarz inequality. This argument goes back to Aronson [4] and has been used in Grigor'yan [33] and followed by Saloff-Coste [61]. To save the length of the paper, we omit it. $\square$

Proof of Proposition 5.6. Let $\rho > 0$ to be fixed. Define

$$F(y,t) = \int\limits_{M \setminus B(o,\rho)} H(x,z,T)H(z,y,t)\,\mathrm{d}\mu(z).$$

Then it is clear that $F(y,t)$ is a positive solution of the heat equation with initial date

$$F(y,0) = \begin{cases} H(x,y,T) & \text{for } y \in M \setminus B(o,\rho), \\ 0 & \text{for } y \in B(o,\rho). \end{cases}$$

Clearly, $F(y,t) \leqslant H(x,y,t+T)$ and hence $F(\cdot,t) \in L^2(\mu)$. Taking

$$g(y,t) = -\frac{d^2(x,y)}{(1+2\delta)T - t}$$

in Lemma 5.7, we can prove that, for any $R > 0$, $\delta > 0$ and $t \leqslant (1+2\delta)T$, we have

$$\int\limits_{B(x,R)} \exp\left(-\frac{d^2(x,y)}{2[(1+2\delta)T - t]}\right) F^2(y,t)\,\mathrm{d}\mu(y) \leqslant \int\limits_{M} \exp\left(-\frac{d^2(x,y)}{2(1+2\delta)T}\right) F^2(y,0)\,\mathrm{d}\mu(y).$$

Taking $t = (1+\delta)T$ and since $F(y,0) = H(x,y,T)1_{M\setminus B(o,\rho)}(y)$, we obtain

$$\mathrm{e}^{\frac{-R^2}{2\delta T}} \int\limits_{B(x,R)} F^2\big(y,(1+\delta)T\big)\,\mathrm{d}\mu(y) \leqslant \int\limits_{M\setminus B(o,\rho)} \exp\left(-\frac{d^2(x,y)}{2(1+2\delta)T}\right) H^2(x,y,T)\,\mathrm{d}\mu(y).$$

Thus

$$\int_{B(x,R)} F^2\big(y,(1+\delta)T\big)\,\mathrm{d}\mu(y) \leqslant \mathrm{e}^{\frac{R^2}{2\delta T}} F(x,T) \sup_{y\in M\setminus B(o,\rho)} \mathrm{e}^{-\frac{d^2(x,y)}{2(1+2\delta)T}}.$$

On the other hand, as $F(y,t)$ is a positive solution to the heat equation $\partial_t u = Lu$, the mean value inequality (Corollary 5.3) applies to $F(y,t)$. Taking $p=2$ in Corollary 5.3 we have

$$F^2(x,T) \leqslant \mu^{-1}\big(B_x(R)\big)\left[\int_{B_x(R)} F^2\big(y,(1+\delta)T\big)\,\mathrm{d}\mu(y)\right](1+\delta)^{m\alpha}\exp\left(\frac{\alpha R^2}{2\delta T} + A\delta T\right).$$

From the above two inequalities we get

$$F(x,T) \leqslant \mu^{-1}\big(B_x(R)\big)(1+\delta)^{m\alpha}\exp\left(\frac{(1+\alpha)R^2}{2\delta T} + 2A\delta T\right) \sup_{y\in M\setminus B(o,\rho)} \mathrm{e}^{-\frac{d^2(x,y)}{2(1+2\delta)T}}.$$

Taking $R^2 = T$ we obtain

$$F(x,T) \leqslant C(m,\alpha,\delta)\mu^{-1}\big(B_x\big(\sqrt{T}\big)\big)\mathrm{e}^{2A\delta T} \sup_{y\in M\setminus B(o,\rho)} \mathrm{e}^{-\frac{d^2(x,y)}{2(1+2\delta)T}}.$$

Now taking $\rho = R/2$, $\delta = 3/4$ and $T = t$, we obtain

$$\begin{aligned} I &\leqslant C\mu^{-1}\big(B\big(x,\sqrt{t}\big)\big)\mathrm{e}^{\frac{3At}{2}} \sup_{y\in M\setminus B(o,R/2)} \mathrm{e}^{\frac{-d^2(x,y)}{5t}} \\ &= C\mu^{-1}\big(B\big(x,\sqrt{t}\big)\big)\mathrm{e}^{\frac{3At}{2} - \frac{(R-2d(x,o))^2}{20t}}. \qquad \square \end{aligned}$$

5.3. Two integral inequalities on derivatives of heat kernel

In the proof of the L^1-Liouville theorem for Δ on a complete Riemannian manifold, Li [41] has used an integral inequality proved by Cheng, Li and Yau [16] for the heat kernel on any complete Riemannian manifold (not necessarily with bounded curvature) which states that: for any $\beta \in \mathbb{N}$ there exists a constant $C(\beta) > 0$ such that and any $x \in M$, the heat kernel $H(x,y,t)$ of Δ satisfies

$$\left|\int_M H(x,y,t)\Delta_y^\beta H(x,y,t)\,\mathrm{d}y\right| \leqslant t^{-\beta}C(\beta)\int_M H^2(x,y,t/2)\,\mathrm{d}y. \tag{5.35}$$

To prove this inequality, Cheng, Li and Yau used the method of eigenfunction expansion to show that the heat kernel $H_i(x,y,t)$ on $B(x,R_i)$ (which is a compact exhaustion of M) with the Dirichlet boundary condition satisfies

$$\left|\int_{B(x,R_i)} H_i(x,y,t)\Delta_y^\beta H_i(x,y,t)\,\mathrm{d}y\right| \leqslant t^{-\beta}C(\beta)\int_{B(x,R_i)} H_i^2(x,y,t/2)\,\mathrm{d}y, \tag{5.36}$$

and then used the fact that the monotone increasing sequence $H_i(x,y,t)$ converges uniformly to $H(x,y,t)$ on compact sets. We would like to point out that, to obtain the uniform convergence, Cheng, Li and Yau [16] has first proved a upper bound estimate of the heat kernel

$$H_i(x,y,t) \leqslant C\big(n,k(2d),\tilde{\delta}(x),T\big)t^{-n/2}\exp\left(\frac{-d^2}{16t}\right) \tag{5.37}$$

for all $t \in [0,T]$ and all $x \in M$. Here, according to the notation used in [16], $d = d(x,y)$, $k(2d)$ denotes the lower bound of the sectional curvature on $B(x,2d)$.

When one deals with a general symmetric diffusion operator $L = \Delta - \nabla\phi\cdot\nabla$, it seems to the author that it is not easy to extend the Cheng–Li–Yau upper bound estimate (5.37) to the Dirichlet heat kernel of $L = \Delta - \nabla\phi\cdot\nabla$ on $B(x,R_i)$, since on the one hand there is no suitable notion of "sectional curvature for L" and on the other hand there should be many preliminary results which one needs to establish in order to prove (5.37).

However, we can still use the eigenfunction expansion method as used in Cheng, Li and Yau [16] for Δ to prove that (5.35) remains valid for the heat kernel $H(x,y,t)$ of L (certainly we need to replace Δ in (5.35) by L) even though we may not easily extend (5.37) in a suitable way. Indeed, letting $H_i(x,y,t)$ denote the heat kernel of L on $B(x,R_i)$ with the Dirichlet boundary condition, then the maximum principle implies that $H_i(x,y,t)$ is a monotone increasing sequence which converges to the heat kernel $H(x,y,t)$ of L on M. Moreover, using the famous result due to Aronson and Nash, $H(x,y,t)$ is locally Hölder continuous. By Dini's theorem, the monotone convergence of $H_i(x,y,t)$ to $H(x,y,t)$ is also uniform on all compact sets. Thus, the eigenfunction expansion method used in the proof of Lemma 7 in Cheng, Li and Yau [16] remains valid for $L = \Delta - \nabla\phi\cdot\nabla$ and for all $\beta \in \mathbb{N}$.

Here, we would like to give a new proof of the generalized Cheng–Li–Yau inequality for the heat kernel of $L = \Delta - \nabla\phi\cdot\nabla$. In a very similar way, this method has been used earlier by Varopoulos [69] and Saloff-Coste [59] in the study of L^∞-bound of time derivatives of the heat kernel of the Laplace–Beltrami operator on complete Riemannian manifolds.

Proposition 5.8. *Let $L = \Delta - \nabla\phi\cdot\nabla$ be a symmetric diffusion operator on a complete Riemannian manifold M with $\phi \in C^2(M)$. Then for any $\beta > 0$ there exists a constant $C(\beta) > 0$ such that for all $x \in M$,*

$$\left|\int_M H(x,y,t)L_y^\beta H(x,y,t)\,\mathrm{d}\mu(y)\right| \leqslant C(\beta)t^{-\beta}\int_M H^2(x,y,t/2)\,\mathrm{d}\mu(y).$$

Proof. Note that $L = \Delta - \nabla\phi\cdot\nabla$ is symmetric with respect to μ. Using the spectral decomposition we have

$$\int_M H(x_0,x,t)L_x^\beta H(x_0,x,t)\,\mathrm{d}\mu(x)=\int \left|L_x^{\beta/2}H(x_0,x,t)\right|^2\mathrm{d}\mu(x).$$

Therefore we need only to prove:

$$\int_M \left|L_x^{\beta/2}H(x,x_0,t)\right|^2\mathrm{d}\mu(x)\leqslant C(\beta)t^{-\beta}\int_M H^2(x,x_0,t/2)\,\mathrm{d}\mu(x).$$

Since $L=\Delta-\nabla\phi\cdot\nabla$ is self-adjoint in $L^2(M,\mu)$, the heat semigroup $P_t=\mathrm{e}^{-tL}$ is analytic in $L^p(M,\mu)$ for all $p\in(1,\infty)$. Hence, for all $\beta>0$ and $p>1$ there exists a constant $C_p(\beta)>0$ such that $f\in C^\infty(M)$,

$$\left\|L^{\beta/2}\mathrm{e}^{-tL}f\right\|_p=\left\|L^{\beta/2}\mathrm{e}^{-tL/2}\mathrm{e}^{-tL/2}f\right\|_p\leqslant C_p(\beta)t^{-\beta/2}\left\|\mathrm{e}^{-tL/2}f\right\|_p.$$

In particular, for $p=2$, we have

$$\left\|L^{\beta/2}\mathrm{e}^{-tL}f\right\|_2^2\leqslant C_2^2(\beta)t^{-\beta}\left\|\mathrm{e}^{-tL/2}f\right\|_2^2. \tag{5.38}$$

(Indeed, using the spectral decomposition, we have

$$\begin{aligned}\left\|L^{\beta/2}\mathrm{e}^{-tL}f\right\|_2^2&=\left\|\int_0^\infty \lambda^{\beta/2}\mathrm{e}^{-\lambda t}\,\mathrm{d}E_\lambda f\right\|_2^2\leqslant C_2^2(\beta)t^{-\beta}\left\|\int_0^\infty \mathrm{e}^{-\lambda t/2}\,\mathrm{d}E_\lambda f\right\|_2^2\\&=C_2^2(\beta)t^{-\beta}\left\|\mathrm{e}^{-tL/2}f\right\|_2^2\end{aligned}$$

providing $\lambda^{\beta/2}\mathrm{e}^{-\lambda t}\leqslant C_2(\beta)t^{-\beta/2}\mathrm{e}^{-\lambda t/2}$. This is true if we take $C_2(\beta):=\sup_{t>0}[t^{\beta/2}\times \mathrm{e}^{-t/2}]$.) Now taking $f_n(y)\to\delta_{x_0}(y)$ in the sense of distribution, as in [69] where $\phi\equiv 0$, we have

$$\lim_{n\to\infty}\int_M\left|\int_M H(x,y,t/2)f_n(y)\mathrm{e}^{-\phi(y)}\,\mathrm{d}y\right|^2\mathrm{d}\mu(x)=\int_M H^2(x,x_0,t/2)\mathrm{e}^{-2\phi(x_0)}\,\mathrm{d}\mu(x).$$

On the other hand, Fatou's lemma yields

$$\begin{aligned}&\int_M\left|L_x^{\beta/2}H(x,x_0,t)\mathrm{e}^{-\phi(x_0)}\right|^2\mathrm{d}\mu(x)\\&\quad=\int_M\lim_{n\to\infty}\left|\int_M L_x^{\beta/2}H(x,y,t)f_n(y)\,\mathrm{d}\mu(y)\right|^2\mathrm{d}\mu(x)\\&\quad\leqslant\liminf_{n\to\infty}\int_M\left|\int_M L_x^{\beta/2}H(x,y,t)f_n(y)\mathrm{e}^{-\phi(y)}\,\mathrm{d}y\right|^2\mathrm{d}\mu(x).\end{aligned}$$

Combing these with the inequality (5.38) we conclude that for $C(\beta) = C_2^2(\beta)$,

$$\int_M \left|L_x^{\beta/2} H(x, x_0, t)\right|^2 \mathrm{d}\mu(x) \leqslant C(\beta) t^{-\beta} \int_M H^2(x, x_0, t/2)\, \mathrm{d}\mu(x). \qquad \square$$

The following result extends another inequality appeared in Cheng, Li and Yau [16] for Δ and will be used in the next section.

Proposition 5.9. *Let $L = \Delta - \nabla\phi \cdot \nabla$ be a diffusion operator on a complete Riemannian manifold M with $\phi \in C^2(M)$. Then for all $o, x \in M$ and all $r > 0$, we have*

$$\int_{B^c(o,3r/4)} \left|\nabla H(y)\right|^2 \mathrm{d}\mu(y) \leqslant \frac{64}{r^2} \int_{B^c(o,r/2)} H^2(y)\, \mathrm{d}\mu(y) + 2 \int_{B^c(o,r/2)} H(y) \left|LH(y)\right| \mathrm{d}\mu(y). \tag{5.39}$$

where $H(y)$ denotes $H(x, y, t)$, $B^c(o, r/2) = M \setminus B(o, r/2)$, etc.

Proof. Let $R > r$. Let η be the cut-off function defined by

$$\eta(y) = \begin{cases} 1 & \text{on } B(o, R) \setminus B(o, 3r/4), \\ 0 & \text{on } M \setminus B(o, 2R), \\ 0 & \text{on } B(o, r/2), \end{cases}$$

and such that $0 \leqslant \eta(y) \leqslant 1$ and $|\nabla\eta(y)| \leqslant 4/r$ for all $y \in M$. Note that the L^2-adjoint of the exterior differential operator d with respect to $\mathrm{d}\mu = \mathrm{e}^{-\phi}\, \mathrm{d}x$ is $d_\phi^* = d^* - i_{\nabla\phi}$, where $i_{\nabla\phi}$ is the interior multiplication by the vector field $X = \nabla\phi$. Therefore

$$\begin{aligned} \int_M \eta^2 |\nabla H|^2\, \mathrm{d}\mu &= \int_M (\mathrm{d}H, \eta^2\, \mathrm{d}H)\, \mathrm{d}\mu = \int_M H d_\phi^*(\eta^2\, \mathrm{d}H)\, \mathrm{d}\mu \\ &= -2 \int_M (H\, \mathrm{d}\eta, \eta\, \mathrm{d}H)\, \mathrm{d}\mu - \int_M \eta^2 H L H\, \mathrm{d}\mu \\ &\leqslant 2 \int |\nabla\eta|^2 H^2\, \mathrm{d}\mu + \frac{1}{2} \int \eta^2 |\nabla H|^2\, \mathrm{d}\mu - \int \eta^2 H L H\, \mathrm{d}\mu. \end{aligned}$$

This yields

$$\int_M \eta^2|\nabla H|^2 \,\mathrm{d}\mu \leqslant 4\int |\nabla\eta|^2 H^2 \,\mathrm{d}\mu - 2\int \eta^2 HLH \,\mathrm{d}\mu$$
$$\leqslant \frac{64}{r^2}\int_{M\setminus B(o,r/2)} H^2 \,\mathrm{d}\mu + 2\int_{M\setminus B(o,r/2)} H|LH| \,\mathrm{d}\mu.$$

Letting $R \to \infty$, we prove the desired inequality. □

Corollary 5.10. *Let $L = \Delta - \nabla\phi \cdot \nabla$ be a diffusion operator on a complete Riemannian manifold M with $\phi \in C^2(M)$. Then for all $o, x \in M$ and $r > 0$, we have*

$$\int_{M\setminus B(o,3r/4)} \left|\nabla H(x,y,t)\right|^2 \mathrm{d}\mu(y)$$
$$\leqslant \left(\int_{M\setminus B(o,r/2)} H^2\right)^{1/2}\left[\frac{64}{r^2}H^{1/2}(x,x,2t) + \frac{2C}{t}H^{1/2}(x,x,t)\right]. \tag{5.40}$$

Proof. By the Cauchy–Schwarz inequality we have

$$\int_{M\setminus B(o,3r/4)} |\nabla H|^2 \leqslant \left(\int_{M\setminus B(o,r/2)} H^2\right)^{1/2}\left[\frac{64}{r^2}\left(\int_M H^2\right)^{1/2} + 2\left(\int_M |LH|^2\right)^{1/2}\right].$$

Combining this with Propositions 5.8 and 5.9, and using $\int_M H^2(x,y,t)\,\mathrm{d}\mu(y) = H(x,x,2t)$, we obtain (5.40). □

6. Integration by parts formula

The following integration by parts formula will play a crucial role in the proofs of the L^1-Liouville theorem and the L^1-uniqueness of heat semigroup.

Theorem 6.1. *Suppose that there exists a constant $C > 0$ such that*

$$Ric + \nabla^2\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n} \geqslant -C\left(1+\rho(x)^2\right), \quad \forall x \in M,$$

where $\rho(x) = d(x,o)$, $o \in M$ is a fixed point. Then, for any non-negative subharmonic function $g \in L^1(M,\mu)$, we have:

$$\int_M L_y H(x,y,t)g(y)\,\mathrm{d}\mu(y) = \int H(x,y,t)Lg(y)\,\mathrm{d}\mu(y). \tag{6.41}$$

To prove this result, we need the following Green formula.

Lemma 6.2. *Let $\Omega \subset M$ be a bounded domain with C^1-boundary, ∂_ν denotes the exterior normal derivative. Then, for any $u, v \in C^2(M)$, we have*:

$$\int_\Omega Lu v \,\mathrm{d}\mu = \int_\Omega u L v + \int_{\partial\Omega} v \partial_\nu u \,\mathrm{d}\mu_\sigma - \int_{\partial\Omega} u \partial_\nu v \,\mathrm{d}\mu_\sigma,$$

where

$$\mathrm{d}\mu_\sigma(y) = \mathrm{e}^{-\phi(y)}\,\mathrm{d}\sigma(y)$$

is the weight area-measure induced by $\mathrm{d}\mu(y) = \mathrm{e}^{-\phi(y)}\,\mathrm{d}y$ *on* $\partial\Omega$.

Proof. By standard Green's formula for the volume measure, we have:

$$\begin{aligned}
\int_\Omega \Delta u\big(v\mathrm{e}^{-\phi}\big)\,\mathrm{d}y &= \int_{\partial\Omega} \partial_\nu u\big(v\mathrm{e}^{-\phi}\big)\,\mathrm{d}\sigma(y) - \int_\Omega \big\langle \nabla u(y), \nabla\big(v(y)\mathrm{e}^{-\phi(y)}\big)\big\rangle\,\mathrm{d}y \\
&= \int_{\partial\Omega} \partial_\nu u v\,\mathrm{d}\mu_\sigma(y) - \int_\Omega \big\langle \nabla u(y), \nabla\big(v(y)\mathrm{e}^{-\phi(y)}\big)\big\rangle\,\mathrm{d}y \\
&= \int_{\partial\Omega} \partial_\nu u v\,\mathrm{d}\mu_\sigma(y) - \int_\Omega \langle \nabla u, \nabla v\rangle\,\mathrm{d}\mu(y) + \int_\Omega \nabla\phi(y)\cdot\nabla u(y) v(y)\,\mathrm{d}\mu(y).
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_\Omega Lu(y)v(y)\,\mathrm{d}\mu(y) &= \int_\Omega \big(\Delta u(y) - \nabla\phi(y)\cdot\nabla u(y)\big)v(y)\mathrm{e}^{-\phi(y)}\,\mathrm{d}y \\
&= \int_{\partial\Omega} v\partial_\nu u\,\mathrm{d}\mu_\sigma - \int_\Omega \langle\nabla u, \nabla v\rangle\,\mathrm{d}\mu(y).
\end{aligned}$$

Similarly

$$\int_\Omega uLv\,\mathrm{d}\mu = \int_{\partial\Omega} u\partial_\nu v\,\mathrm{d}\mu_\sigma - \int_\Omega \langle\nabla u, \nabla v\rangle\,\mathrm{d}\mu(y).$$

The desired result follows from the above identities. □

Proof of Theorem 6.1. Applying the Green formula on $B(o, R)$, we have:

$$
\begin{aligned}
&\left| \int\limits_{B(o,R)} L_y H(x,y,t) g(y)\,\mathrm{d}\mu(y) - \int\limits_{B(o,R)} H(x,y,t) Lg(y)\,\mathrm{d}\mu(y) \right| \\
&\quad = \left| \int\limits_{\partial B(o,R)} \frac{\partial H}{\partial \nu}(x,y,t) g(y)\,\mathrm{d}\mu_{\sigma,R}(y) - \int\limits_{\partial B(o,R)} H(x,y,t) \frac{\partial g}{\partial \nu}\,\mathrm{d}\mu_{\sigma,R}(y) \right| \\
&\quad \leqslant \int\limits_{\partial B(o,R)} \left|\nabla_y H(x,y,t)\right| g(y)\,\mathrm{d}\mu_{\sigma,R}(y) + \int\limits_{\partial B(o,R)} H(x,y,t) |\nabla g|(y)\,\mathrm{d}\mu_{\sigma,R}(y),
\end{aligned}
$$

where $\mu_{\sigma,R}$ denotes the weight area-measure induced by μ on $\partial B(o,R)$. It remains to show that the above two boundary integrals tend to zero as $R \to \infty$.

Step 1. Based on the generalized Laplacian comparison theorem and the Bishop–Cheeger–Gromov type volume comparison theorem with respect to μ, using a similar argument as used in the proof of Theorem 2.1 in Li and Schoen [42],we can prove the following mean value inequality for L-subharmonic function on every geodesic ball $B(o,R)$, i.e., if we let $-K(R)$ be the lower bound of the Bakry–Emery Ricci curvature $Ric_{m,n}(L)$ on $B(o,R)$, then for some constants C and α depending only on m, we have

$$
\sup_{B(o,R)} g(x) \leqslant C \frac{\mathrm{e}^{\alpha\sqrt{K(R)}R}}{\mu(B(o,2R))} \int\limits_{B(o,2R)} g(y)\,\mathrm{d}\mu(y). \tag{6.42}
$$

Therefore, under the assumption $Ric_{m,n}(L) \geqslant -C(1+\rho(x)^2)$ we have

$$
\sup_{B(o,R)} g(x) \leqslant \frac{C\mathrm{e}^{\alpha R^2}}{\mu(B(o,2R))} \|g\|_{L^1(\mu)}.
$$

Let $\phi(y) = \phi(\rho(y))$ be a cut-off function such that $0 \leqslant \phi \leqslant 1$, $|\nabla\phi| \leqslant \sqrt{3}$ and

$$
\phi(\rho(y)) = \begin{cases} 1 & \text{on } B(o,R+1) \setminus B(o,R), \\ 0 & \text{on } B(o,R-1) \cup (M \setminus B(o,R+2)). \end{cases}
$$

By the L-subharmonicity of g and the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
0 &\leqslant \int\limits_M \phi^2(y) g(y) Lg(y)\,\mathrm{d}\mu(y) = -\int\limits_M \nabla\big(\phi^2(y) g(y)\big) \nabla g(y)\,\mathrm{d}\mu(y) \\
&= -2\int\limits_M \phi g \langle \nabla\phi, \nabla g\rangle\,\mathrm{d}\mu - \int\limits_M \phi^2 |\nabla g|^2\,\mathrm{d}\mu \\
&\leqslant 2\int\limits_M |\nabla\phi|^2 g^2\,\mathrm{d}\mu - \frac{1}{2}\int\limits_M \phi^2 |\nabla g|^2\,\mathrm{d}\mu.
\end{aligned}
$$

Thus

$$\int\limits_{B(o,R+1)\setminus B(o,R)} \left|\nabla g(y)\right|^2 \mathrm{d}\mu(y) \leqslant 4\int\limits_M |\nabla\phi|^2 g^2\,\mathrm{d}\mu \leqslant 12 \int\limits_{B(o,R+2)} g^2\,\mathrm{d}\mu$$
$$\leqslant 12\|g\|_{L^1(\mu)} \sup_{B(o,R+2)} g(y)$$
$$\leqslant \frac{C\mathrm{e}^{\alpha(R+2)^2}\|g\|^2_{L^1(\mu)}}{\mu(B(o,2R+4))},$$

from which and using the Cauchy–Schwarz inequality we have

$$\int\limits_{B(o,R+1)\setminus B(o,R)} |\nabla g|\,\mathrm{d}\mu \leqslant \left(\int\limits_{B(o,R+1)\setminus B(o,R)} |\nabla g|^2\,\mathrm{d}\mu\right)^{1/2}\sqrt{B(o,R+1)\setminus B(o,R)}$$
$$\leqslant \left(\int\limits_{B(o,R+1)\setminus B(o,R)} |\nabla g|^2\,\mathrm{d}\mu\right)^{1/2}\sqrt{\mu\big(B(o,2R+4)\big)}.$$

Hence

$$\int\limits_{B(o,R+1)\setminus B(o,R)} |\nabla g|\,\mathrm{d}\mu \leqslant C\|g\|_{L^1(\mu)}\mathrm{e}^{\alpha R^2}. \tag{6.43}$$

Step 2. By Lott [47], under the curvature-dimension condition for L, the Bishop–Cheeger–Gromov type relative volume comparison theorem holds for μ. Hence

$$\mu\big(B_x\big(\sqrt{t}\big)\big) \leqslant \mu\big(B_y\big(d(x,y)+\sqrt{t}\big)\setminus B_y\big(d(x,y)-\sqrt{t}\big)\big)$$
$$\leqslant \mu\big(B_y\big(\sqrt{t}\big)\big)\frac{V(B(K(R),d(x,y)+\sqrt{t}))-V(B(K(R),d(x,y)-\sqrt{t}))}{V(B(K(R),\sqrt{t}))}$$
$$\leqslant \mu\big(B_y\big(\sqrt{t}\big)\big)\frac{V(B(K(R),d(x,y)+\sqrt{t}))}{V(B(K(R),\sqrt{t}))}$$
$$\leqslant \mu\big(B_y\big(\sqrt{t}\big)\big)\frac{C\exp(\sqrt{(m-1)K(R)}\,(d(x,y)+\sqrt{t})}{t^{m/2}},$$

where $V(B(K(R),\sqrt{t}))$ denotes the volume of the geodesic ball of radius $\sqrt{t}$ in the m-dimensional hyperbolic model space $H(-\frac{K(R)}{m-1})$ of constant sectional curvature $-\frac{K(R)}{m-1}$. Combining the above inequality with (5.33), we obtain:

$$H(x,y,t) \leqslant \frac{C}{\mu(B_x(\sqrt{t}))t^{m/4}}\exp\left(\frac{-d^2(x,y)}{5t}+\alpha\big(R^2+R^{-2}\big)t+\alpha R\big(d(x,y)+\sqrt{t}\big)\right). \tag{6.44}$$

Step 3. Combining (6.43) with (6.44) we obtain

$$\begin{aligned}
J_1 &:= \int\limits_{B(o,R+1)\setminus B(o,R)} H(x,y,t)|\nabla g|(y)\,\mathrm{d}\mu(y)\\
&\leqslant \Big(\sup_{y\in B(o,R+1)\setminus B(o,R)} H(x,y,t)\Big)\int\limits_{B(o,R+1)\setminus B(o,R)} |\nabla g|\,\mathrm{d}\mu\\
&\leqslant C\|g\|_1 \mathrm{e}^{\alpha R^2}\mu^{-1}\big(B_x\big(\sqrt{t}\big)\big)t^{-m/4}\\
&\quad\times\exp\bigg(\frac{-(R-d(o,x))^2}{5t}+\alpha\big(R^2+R^{-2}\big)t+\alpha R\big(R+1+d(o,x)+\sqrt{t}\big)\bigg).
\end{aligned}$$

Note that

$$\begin{aligned}
&\alpha R^2-\frac{(R-d(o,x))^2}{5t}+\alpha\big(R^2+R^{-2}\big)t+\alpha R\big(R+1+d(o,x)+\sqrt{t}\big)\\
&\quad=\bigg(2\alpha-\frac{1}{5t}+\alpha t\bigg)R^2+\frac{2Rd(o,x)}{5t}-\frac{d^2(o,x)}{5t}+\alpha Rd(o,x)\\
&\qquad+\alpha R\big(1+\sqrt{t}\big)+\alpha R^{-2}t\\
&\quad\leqslant\bigg(2\alpha+\alpha t-\frac{1}{5t}\bigg)R^2+\frac{R^2}{10t}+\frac{1}{5t}d^2(o,x)+\frac{\alpha}{2}R^2+\frac{\alpha}{2}d^2(o,x)+\frac{\alpha}{2}R^2\\
&\qquad+\frac{\alpha}{2}\big(\sqrt{t}+1\big)^2+\alpha R^{-2}t\\
&\quad\leqslant\bigg(3\alpha+\alpha t-\frac{1}{10t}\bigg)R^2+\bigg(\frac{1}{5t}+\frac{\alpha}{2}\bigg)d^2(o,x)+\alpha(t+1)+\alpha R^{-2}t.
\end{aligned}$$

Thus, for T sufficiently small and for all $t\in(0,T)$ there exist some constants $\beta>0$, $C_1>0$, $C_2>0$ such that

$$\begin{aligned}
J_1 &\leqslant C\|g\|_1\mu^{-1}\big(B_x\big(\sqrt{t}\big)\big)t^{-m/4}\exp\big(-\beta R^2+C_1\big(1+t^{-1}\big)d^2(o,x)\\
&\quad+C_2\big(1+R^{-2}\big)t\big).
\end{aligned}\tag{6.45}$$

Step 4. By (6.42) and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
J_2 &:= \int\limits_{B(o,R+1)\setminus B(o,R)} \big|\nabla H(x,y,t)\big|g(y)\,\mathrm{d}\mu(y)\\
&\leqslant \sup_{y\in B(o,R+1)\setminus B(o,R)} g(y)\int\limits_{B(o,R+1)\setminus B(o,R)} \big|\nabla H(x,y,t)\big|\,\mathrm{d}\mu(y)
\end{aligned}$$

$$\leqslant \frac{C\|g\|_1 e^{\alpha(R+1)^2}}{\mu(B(o,2R+2))}\sqrt{\mu\big(B(o,R+1)\setminus B(o,R)\big)}$$
$$\times\Bigg[\int\limits_{B(o,R+1)\setminus B(o,R)} \big|\nabla H(x,y,t)\big|^2\,d\mu(y)\Bigg]^{1/2}.$$

By (5.40), (5.33) and (5.34), we have

$$\int\limits_{M\setminus B(o,R)} \big|\nabla H(x,y,t)\big|^2\,d\mu(y)$$
$$\leqslant\Bigg(\int\limits_{M\setminus B(o,R/2)} H^2(x,y,t)\,d\mu(y)\Bigg)^{1/2}\big[64R^{-2}H^{1/2}(x,x,2t)+2Ct^{-1}H^{1/2}(x,x,t)\big]$$
$$\leqslant C\big[R^{-2}\mu^{-1/2}\big(B\big(x,\sqrt{2t}\,\big)\big)+t^{-1}\mu^{-1/2}\big(B\big(x,\sqrt{t}\,\big)\big)\big]e^{\alpha\varepsilon(K(R)+R^{-2})t}$$
$$\times\,\mu^{-1/2}\big(B\big(x,\sqrt{t}\,\big)\big)e^{\frac{3At}{4}-\frac{(R-2d(x,o))^2}{40t}}$$
$$\leqslant C\big[R^{-2}+t^{-1}\big]\mu^{-1}\big(B\big(x,\sqrt{t}\,\big)\big)e^{C(R^2+R^{-2})t-\frac{(R-2d(x,o))^2}{40t}}.$$

Therefore

$$J_2\leqslant C\|g\|_1 e^{\alpha(R+1)^2}\mu^{-1}\big(B(o,2R+2)\big)\mu^{1/2}\big(B(o,R+1)\big)\big[R^{-2}+t^{-1}\big]^{1/2}$$
$$\times\,\mu^{-1/2}\big(B\big(x,\sqrt{t}\,\big)\big)\exp\bigg(\frac{-(R-2d(o,x))^2}{80t}+C\big(R^2+R^{-2}\big)t\bigg)$$
$$\leqslant\frac{C\|g\|_1[R^{-2}+t^{-1}]^{1/2}e^{2\alpha R^2}}{\mu^{1/2}(B(o,2R+2))}\mu^{-1/2}\big(B\big(x,\sqrt{t}\,\big)\big)$$
$$\times\exp\bigg(\frac{-(R-2d(o,x))^2}{80t}+C\big(R^2+R^{-2}\big)t\bigg).$$

Similar to the case of J_1, when $T>0$ is small enough, there exist $\beta>0$, $C_1>0$, $C_2>0$ such that for all $t\in(0,T)$ and $R>0$, we have:

$$J_2\leqslant\frac{C\|g\|_1[R^{-2}+t^{-1}]^{1/2}}{\mu^{1/2}(B(o,2R+2))}\mu^{-1/2}\big(B\big(x,\sqrt{t}\,\big)\big)$$
$$\times\exp\big(-\beta R^2+C_1\big(1+t^{-1}\big)d^2(o,x)+C_2\big(1+R^{-2}\big)t\big).\tag{6.46}$$

Step 5. By the co-area formula for all $f \in C_0^\infty(M)$ we have

$$\int_{B(o,R+1)\setminus B(o,R)} f(y)\,\mathrm{d}\mu(y) = \int_R^{R+1} \left[\int_{\partial B(o,r)} f(y)\,\mathrm{d}\mu_{\sigma,r}(y) \right] \mathrm{d}r,$$

where $\mu_{\sigma,r}$ denotes the weight area-measure induced by μ on $\partial B(o,r)$. Therefore, the mean value theorem yields that for any $R > 0$ there exists $\overline{R} \in (R, R+1)$ such that

$$\begin{aligned} J :&= \int_{\partial B(o,\overline{R})} \left(\left|\nabla H(x,y,t)\right| g(y) + H(x,y,t)\left|\nabla g(y)\right|\right) \mathrm{d}\mu_{\sigma,\overline{R}}(y) \\ &= \int_{B(o,R+1)\setminus B(o,R)} \left(\left|\nabla H(x,y,t)\right| g(y) + H(x,y,t)\left|\nabla g(y)\right|\right) \mathrm{d}\mu(y). \end{aligned}$$

By (6.45) and (6.46) we obtain

$$\begin{aligned} J &\leqslant C\|g\|_1 \mu^{-1}\left(B_x\left(\sqrt{t}\right)\right) t^{-m/4} \exp\left(-\beta R^2 + C_1\left(1+t^{-1}\right)d^2(o,x) + C_2\left(1+R^{-2}\right)t\right) \\ &\quad + \frac{C\|g\|_1[R^{-2}+t^{-1}]^{1/2}}{\mu^{1/2}(B(o,2R+2))} \mu^{-1/2}\left(B\left(x,\sqrt{t}\right)\right) \\ &\quad \times \exp\left(-\beta R^2 + C_1\left(1+t^{-1}\right)d^2(o,x) + C_2\left(1+R^{-2}\right)t\right) \\ &\leqslant C\|g\|_1 \left[t^{-m/4}\mu^{-1}\left(B_x\left(\sqrt{t}\right)\right) + \frac{[R^{-2}+t^{-1}]^{1/2}}{\mu^{1/2}(B(o,2R+2))} \mu^{-1/2}\left(B\left(x,\sqrt{t}\right)\right) \right] \\ &\quad \times \exp\left(-\beta R^2 + C_1\left(1+t^{-1}\right)d^2(o,x) + C_2\left(1+R^{-2}\right)t\right). \end{aligned}$$

Hence, for all $t \in (0,T)$ and all $x \in M$, J tends to zero as $\overline{R}$ tends to infinity. This proves that the integration by parts formula holds for $t \in (0,T)$ and all $x \in M$.

Step 6. Using the semigroup property we have

$$\begin{aligned} \frac{\partial}{\partial(s+t)} \mathrm{e}^{(s+t)L} g &= \frac{\partial}{\partial t}\left(\mathrm{e}^{sL}\mathrm{e}^{tL} g\right) = \mathrm{e}^{sL} \frac{\partial}{\partial t}\left(\mathrm{e}^{tL} g\right) \\ &= \mathrm{e}^{sL}\mathrm{e}^{tL}(Lg) = \mathrm{e}^{(s+t)L}(Lg). \end{aligned}$$

Thus, the integration by parts formula holds for all $t > 0$ and all $x \in M$. $\square$

7. L^1-Liouville theorem and L^1-uniqueness

In this section we prove the L^1-Liouville theorem and the L^1-uniqueness of heat semigroup.

7.1. L^1-Liouville theorem

Theorem 7.1. *Suppose that there exists a constant $C > 0$ such that*

$$Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geqslant -C\big(1 + \rho(x)^2\big), \quad \forall x \in M,$$

where $\rho(x) = d(x, o)$, $o \in M$ is a fixed point. Then

(1) *Every $L^1(\mu)$-integrable non-negative L-subharmonic function on M must be identically constant.*
(2) *Every $L^1(\mu)$-integrable L-harmonic function on M must be identically constant.*

Proof. Let g be a non-negative L^1-integrable L-subharmonic function on M. Define:

$$\big(e^{tL}g\big)(x) = \int_M H(x, y, t)g(y)\,d\mu(y).$$

Then

$$\frac{\partial}{\partial t}\big(e^{tL}g\big)(x) = \int_M \frac{\partial}{\partial t}H(x, y, t)g(y)\,d\mu(y) = \int_M L_y H(x, y, t)g(y)\,d\mu(y).$$

By Theorem 6.1 we have

$$\frac{\partial}{\partial t}e^{tL}g = e^{tL}Lg \geqslant 0.$$

Therefore, $e^{tL}g(x)$ is increasing in t.

On the other hand, under the assumption of the theorem, we have $\int H(x, y, t)\,d\mu(x) = 1$ (see Theorem 1.4). By Fubini's theorem we have

$$\begin{aligned}\big\|e^{tL}g\big\|_{L^1(\mu)} &= \int e^{tL}g(x)\,d\mu(x) = \iint H(x, y, t)g(y)\,d\mu(y)\,d\mu(x)\\ &= \int g(y)\bigg(\int H(x, y, t)\,d\mu(x)\bigg)d\mu(y) = \int g(y)\,d\mu(x) = \|g\|_{L^1(\mu)}.\end{aligned}$$

Hence $e^{tL}g(x) = g(x)$ for all $x \in M$ and $t \geqslant 0$. This implies that g must be a harmonic function and $g_a := g \wedge a$ is a non-negative L^1-integrable L-superharmonic function for any non-negative constant $a \geqslant 0$. By K.T. Sturm's generalization of Grigor'yan's result (see [31,33,66]), the conservativeness of L-diffusion process implies that all non-negative L^1-integrable L-superharmonic functions must be constant. Hence, for all $a \geqslant 0$, g_a is a constant. This can only be true provided that g is a constant. Thus, any non-negative L^1-integrable L-subharmonic function must be constant.

If u is a L^1-integrable L-harmonic function, then $g = |u|$ is non-negative L^1-integrable L-subharmonic. Applying the above result to $g = |u|$, we see that $|u|$ must be constant. Since M is connected and u is continuous, we conclude that u must be constant. □

7.2. L^1-uniqueness of heat semigroup

Theorem 7.2. *Suppose that there exists a constant $C > 0$ such that*

$$Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geqslant -C\big(1 + \rho(x)^2\big), \quad \forall x \in M,$$

where $\rho(x) = d(x, o)$, $o \in M$ is a fixed point. Then

(1) *Let $v(x,t)$ be a non-negative function defined on $M \times \mathbb{R}^+$ satisfying*

$$\left(L - \frac{\partial}{\partial t}\right)v(x,t) \geqslant 0, \qquad \int\limits_M v(x,t)\,\mathrm{d}\mu(x) < +\infty$$

for all $t > 0$, and

$$\lim_{t\to 0}\int\limits_M v(x,t)\,\mathrm{d}\mu(x) = 0.$$

Then $v(x,t) = 0$ for all $(x,t) \in M \times \mathbb{R}^+$.

(2) *Every $L^1(\mu)$-integrable solution of the heat equation $\partial_t u = Lu$ is uniquely determined by its initial date $u(0,\cdot) \in L^1(\mu)$.*

Proof. For the completeness of the paper we follow the method used in Li [41] to give a proof of the theorem. For $\varepsilon > 0$, let $v_\varepsilon(x) = v(x,\varepsilon)$. Define $\mathrm{e}^{tL}v_\varepsilon(x) = \int_M H(x,y,t)v_\varepsilon(y)\,\mathrm{d}\mu(y)$, and $F_\varepsilon(x,t) = \max\{0, v(x, t+\varepsilon) - \mathrm{e}^{tL}v_\varepsilon(x)\}$. Then $\lim_{t\to 0} F_\varepsilon(x,t) = 0$ and

$$(L - \partial_t)F_\varepsilon(x,t) \geqslant 0.$$

Let $T > 0$ be fixed. Define $f(x) = \int_0^T F_\varepsilon(x,t)\,\mathrm{d}t$. Then

$$Lf(x) = \int\limits_0^T LF_\varepsilon(x,t)\,\mathrm{d}t \geqslant \int\limits_0^T \frac{\partial}{\partial t}F_\varepsilon(x,t)\,\mathrm{d}t = F_\varepsilon(x,T) \geqslant 0. \tag{7.47}$$

By Fubini's theorem and using the fact that $P_t = \mathrm{e}^{tL}$ is conservative, we have:

$$
\begin{aligned}
\|f\|_{L^1(\mu)} &= \int_M f(x)\,\mathrm{d}\mu(x) = \int_0^T\int_M F_\varepsilon(x,t)\,\mathrm{d}\mu(x)\,\mathrm{d}t \\
&\leqslant \int_0^T\int_M \bigl|v(x,t+\varepsilon) - \mathrm{e}^{tL}v_\varepsilon(x)\bigr|\,\mathrm{d}\mu(x)\,\mathrm{d}t \\
&\leqslant \int_0^T\int_M v(x,t+\varepsilon)\,\mathrm{d}\mu(x)\,\mathrm{d}t + \int_0^T\int_M \mathrm{e}^{tL}v_\varepsilon(x)\,\mathrm{d}\mu(x)\,\mathrm{d}t \\
&= \int_0^T\int_M v(x,t+\varepsilon)\,\mathrm{d}\mu(x)\,\mathrm{d}t + \int_0^T\int_M\int_M H(x,y,t)v_\varepsilon(y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,\mathrm{d}t \\
&= \int_0^T\int_M v(x,t+\varepsilon)\,\mathrm{d}\mu(x)\,\mathrm{d}t + T\int_M v_\varepsilon(y)\,\mathrm{d}\mu(y).
\end{aligned}
$$

Hence $\|f\|_{L^1(\mu)} < \infty$. Therefore, f is a non-negative L^1-integrable subharmonic function. By the L^1-Liouville theorem (i.e., Theorem 7.1), f must be constant. Combining this with (7.47), we have $0 = Lf(x) \geqslant F_\varepsilon(x,T) \geqslant 0$. Hence $F_\varepsilon(x,T) \equiv 0$ for all $x \in M$ and $T > 0$. Therefore

$$\mathrm{e}^{tL}v_\varepsilon(x) \geqslant v(x,t+\varepsilon). \tag{7.48}$$

Applying the upper bound estimate (6.44) of the heat kernel $H(x,y,t)$ and setting $R = 1 + 2d(x,y)$, we have

$$
\begin{aligned}
\mathrm{e}^{tL}v_\varepsilon(x) &\leqslant Ct^{-m/4}\mu^{-1}\bigl(B_x(\sqrt{t})\bigr) \\
&\quad \times \int_M \left[\exp\left(\frac{-d^2(x,y)}{5t} + \alpha t\bigl(1 + d^2(x,y)\bigr)\right)v(y,\varepsilon)\right]\mathrm{d}\mu(y).
\end{aligned}
$$

Hence there exists a sufficiently small $t_0 > 0$ such that for all $0 < t \leqslant t_0$, we have:

$$\mathrm{e}^{tL}v_\varepsilon(x) \leqslant Ct^{-m/4}\mu^{-1}\bigl(B_x(\sqrt{t})\bigr)\int_M v(y,\varepsilon)\,\mathrm{d}\mu(y).$$

Therefore for all $x \in M$ and for all $0 < t \leqslant t_0$,

$$\lim_{\varepsilon\to 0}\mathrm{e}^{tL}v_\varepsilon(x) \leqslant Ct^{-m/4}\mu^{-1}\bigl(B_x(\sqrt{t})\bigr)\lim_{\varepsilon\to 0}\int_M v(y,\varepsilon)\,\mathrm{d}\mu(y) = 0.$$

On the other hand, for $t > t_0$ setting $t = nt_0 + a$, where $a \in (0, t_0)$, by induction and the semigroup property, we have for all $x \in M$,

$$\lim_{\varepsilon\to 0} \mathrm{e}^{tL} v_\varepsilon(x) = \lim_{\varepsilon\to 0} \mathrm{e}^{nt_0L}\big(\mathrm{e}^{aL} v_\varepsilon\big)(x) = \mathrm{e}^{t_0L}\Big(\lim_{\varepsilon\to 0} \mathrm{e}^{(n-1)t_0}\mathrm{e}^{aL} v_\varepsilon\Big)(x)$$
$$= \mathrm{e}^{nt_0L} \lim_{\varepsilon\to 0} \mathrm{e}^{aL} v_\varepsilon(x) = 0.$$

Hence $\lim_{\varepsilon\to 0} \mathrm{e}^{tL} v_\varepsilon(x) = 0$ for all $x \in M$ and $t > 0$. Combining this with (7.48), we have $v(x,t) \leqslant 0$. As $v(x,t)$ is non-negative we obtain $v(x,t) \equiv 0$. This proves (1).

Suppose that $u_1(x,t), u_2(x,t)$ are two $L^1(\mu)$-integrable solutions of the heat equation $\partial_t u = Lu$ with the initial data $u(\cdot,0) \in L^1(M,\mu)$. Applying this above result to $v(x,t) = |u_1(x,t) - u_2(x,t)|$, we see that $v(x,t) \equiv 0$ and hence the heat semigroup is uniquely determined by $u(\cdot,0)$ in $L^1(M,\mu)$. The proof of (2) is finished. □

8. Applications and some further remarks

8.1. Uniqueness of L-invariant measure

In this subsection we give an application of the strong Liouville theorem (i.e., Theorem 1.3) in the study of the problem of uniqueness of L-invariant measure on complete Riemannian manifolds. Recall that a Borel measure ν on M is an invariant measure of L if and only if $L^*\nu = 0$ holds in the sense of distribution, that is, for all $f \in C_0^\infty(M)$, we have

$$\int_M Lf \,\mathrm{d}\nu = 0.$$

In the case where M is a compact Riemannian manifold, a famous theorem due to Kolmogorov [40] (see also Ikeda and Watanabe [37, Proposition 4.5, p. 293]) says that for any elliptic operator of the form $A = \Delta + b$ (where b is a smooth vector field on M), an invariant measure $\nu(\mathrm{d}x)$ of A-diffusion exists and is unique up to a multiplicative constant, moreover $\nu(\mathrm{d}x)$ must be of the form $v(x)\,\mathrm{d}x$ for some $v \in C^\infty(M)$. Hence, in the case where M is a compact Riemannian manifold, the invariant measure of $L = \Delta - \nabla\phi\cdot\nabla$ (which is symmetric on $L^2(M,\mu)$) is unique up to a multiplicative constant and must be of the form $c\mu$ for some constant $c \in \mathbb{R}^+$. The following result gives a natural sufficient condition for the uniqueness of L-invariant measure on complete non-compact Riemannian manifolds.

Theorem 8.1. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $m \geqslant n$ such that*

$$Ric(x) + \nabla^2\phi(x) - \frac{\nabla\phi(x)\otimes\nabla\phi(x)}{m-n} \geqslant 0, \quad \forall x \in M.$$

Then the positive invariant measure of L is unique up to a multiplicative constant and must be of the form $c\mu$, $c \in \mathbb{R}^+$.

Proof. Let ν be an invariant measure of L. Then $L^*\nu = 0$ holds in the sense of distribution. The elliptic regularity (i.e., Weyl's lemma) implies that the restriction of ν on any relative compact open set $U \subset M$ must be of the form $\nu(\mathrm{d}x) = v(x)\,\mathrm{d}x$, where $v \in C^2(U)$. By the standard argument based on the partitions of unity on any complete Riemannian manifold, we conclude that ν must be of the form $\mathrm{d}\nu(x) = v(x)\,\mathrm{d}x$, $v \in C^2(M)$. Therefore, there exists a function $u \in C^2(M)$ such that $\nu = u\mu$. Note that, for all $f \in C_0^\infty(M, \mathbb{R})$, $\int Lf\,\mathrm{d}\nu = \int Lfu\,\mathrm{d}\mu = \int fLu\,\mathrm{d}\mu$. Therefore $L^*\nu = 0$ if and only if $Lu = 0$. Hence, the uniqueness of positive L-invariant measure is equivalent to the strong Liouville theorem for L. Theorem 8.1 follows from Theorem 1.3. □

8.2. L^1-uniqueness of the intrinsic Schrödinger operator

Recall that the diffusion operator L in $L^2(M, \mu)$ is unitary equivalent to the Schrödinger operator $H = \Delta - (|\nabla\phi|^2/4 - \Delta\phi/2)$ in $L^2(M, \mathrm{d}x)$. Using this isomorphism, we have already seen in Section 1.3 that $P_t = \mathrm{e}^{tL}$ is $L^1(M, \mathrm{e}^{-\phi(x)}\,\mathrm{d}x)$-unique if and only if $Q_t = \mathrm{e}^{tH}$ is $L^1(M, \mathrm{e}^{-\phi(x)/2}\,\mathrm{d}x)$-unique. This observation and Theorem 7.2 lead us to state the following:

Theorem 8.2. *Suppose that there exists a constant $C > 0$ such that*

$$Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geqslant -C\big(1 + \rho(x)^2\big), \quad \forall x \in M,$$

where $\rho(x) = d(x, o)$, $o \in M$ is a fixed point. Then the Cauchy problem of the heat equation $\partial_t u = Hu$ is well-posed in $L^1(M, \mathrm{e}^{-\phi/2}\,\mathrm{d}x)$, where $H = \Delta - (|\nabla\phi|^2/4 - \Delta\phi/2)$. That is, every $L^1(\mathrm{e}^{-\phi/2}\,\mathrm{d}x)$-integrable solution of $\partial_t u = Hu$ is uniquely determined by its initial data $u(0, \cdot) \in L^1(M, \mathrm{e}^{-\phi(x)/2}\,\mathrm{d}x)$.

8.3. Example: Ornstein–Uhlenbeck operator

Now we consider the Ornstein–Uhlenbeck operator $L = \Delta - x \cdot \nabla$ on the finite dimensional Gaussian space $(\mathbb{R}^n, \mu)$ with $\mu(\mathrm{d}x) = \mathrm{e}^{-\|x\|^2/2}\,\mathrm{d}x$. In this case, the Bakry–Emery Ricci curvature $Ric_{m,n}(L)$ is given by

$$Ric_{m,n}(L)(x) = \left(1 - \frac{\|x\|^2}{m-n}\right)I, \quad x \in \mathbb{R}^n,$$

in particular, $Ric(L) = Ric_{\infty,n}(L) = I$, where I denotes the identity transformation on $\mathbb{R}^n$. The well-known Mehler formula implies that $P_t = \mathrm{e}^{tL}$ is conservative and the strong Liouville theorem holds for non-negative harmonic function $Lu = 0$. Notice that for any finite dimensional Ornstein–Uhlenbeck operator we have

$$Ric_{m,n}(L)(x) \geqslant -C\big(1 + \|x\|^2\big).$$

By Theorems 7.1 and 7.2, the $L^1(\mu)$-Liouville theorem holds for the solutions of $Lu = 0$ and the heat semigroup $P_t = e^{tL}$ is $L^1(\mu)$-unique. Moreover, $Ric_{m,n}(L)$ satisfies the assumptions required in Theorems 1.4 and 1.5. Therefore, $P_t = e^{tL}$ is a Markovian semigroup and has the C_0-diffusion property. Moreover, the Ornstein–Uhlenbeck operator $L = \Delta - x \cdot \nabla$ which is symmetric on $L^2(\mathbb{R}^n, \mu)$ is unitary equivalent to the harmonic oscillate operator $H = \Delta - (\|x\|^2/4 - n)$ which is symmetric on $L^2(\mathbb{R}^n, dx)$. By Theorem 8.2, we conclude that the heat semigroup $Q_t = e^{tH}$ is $L^1(e^{-\|x\|^2/4}\, dx)$-unique. Indeed, as it will be explained in Section 8.5 below, $P_t = e^{tL}$ is $L^p(\mu)$-unique and $Q_t = e^{tH}$ is $L^p(e^{(p-2)\|x\|^2/4}\, dx)$-unique for all $p \in [1, \infty)$.

8.4. Riesz transforms for ultracontractive diffusion operators

In [7,44], the conservativeness plays a crucial role in the study of the L^p-boundedness of Riesz transforms associated with symmetric diffusion operators. By Theorem 2.2 in [44], if $L = \Delta - \nabla\phi \cdot \nabla$ is a conservative diffusion operator satisfying the utracontractivity property in the sense that

$$\|P_t f\|_{L^\infty} \leqslant \frac{A}{t^{l/2}} \|f\|_{L^1(\mu)}, \quad \forall f \in C_c^\infty(M),\ t \in (0, 1),$$

where $l = \dim(L) > 0$ (is called the ultracontractive dimension of L) and $A > 0$ are some constants and if there exist two constants $\varepsilon > 0$ and $B \geqslant 0$ such that the lowest eigenvalue $\lambda_{\min}(x)$ of the Bakry–Emery Ricci curvature $Ric(L) = Ric + \nabla^2\phi$ satisfies the integrability condition

$$\big(\lambda_{\min}(x) + B\big)^- \in L^{l/2+\varepsilon}(M, \mu),$$

then for all $p \geqslant 2$, the Riesz transforms $R_a(L) = \nabla(a - L)^{-1/2}$ is bounded in $L^p(M, \mu)$.

Assuming that

$$Ric_{m,n}(L)(x) = Ric(L)(x) - \frac{\nabla\phi(x) \otimes \nabla\phi(x)}{m - n} \geqslant -C\big(1 + \rho^2(x)\big),$$

where $\rho(x) = d(o, x)$. Then $\mu(B(o, r)) \leqslant C_1 e^{C_2 r^2}$ for all $r > 0$. Let $u \in C_b(M) \cap C^2(M)$. By Grigor'yan's criterion, the diffusion operator $\tilde{L} = L + \nabla u \cdot \nabla$ is still conservative since $\tilde{\mu}(B(o, r)) \leqslant \widetilde{C}_1 e^{C_2 r^2}$ for all $r > 0$, where $d\tilde{\mu}(x) = e^{-u(x)}\, d\mu(x)$, $\widetilde{C}_1 = C_1 e^{-\min u}$. Since u is bounded, we can prove that if L is a ultracontractive symmetric diffusion operator in $L^2(M, \mu)$ with the dimension $\dim(L) \geqslant 1$, then $\tilde{L}$ is a ultracontractive symmetric diffusion operator in $L^2(M, \tilde{\mu})$ with the same dimension $\dim(\tilde{L}) = \dim(L)$. That is, $\widetilde{P}_t = e^{t\tilde{L}}$ satisfies the ultracontractivity property with $\dim(\tilde{L}) = \dim(L)$. Hence, using Theorem 2.2 in [44] we obtain the following:

Proposition 8.3. *Let $L = \Delta - \nabla\phi \cdot \nabla$ be a ultracontractive diffusion operator on a complete Riemannian manifold (M, g) with $\dim(L) = l \geqslant 1$, $Ric_{m,n}(L) \geqslant -C(1 + \rho^2(x))$.*

Then for all $u \in C_b(M) \cap C^2(M)$, the Riesz transform associated with the diffusion operator $\tilde{L} = L + \nabla u \cdot \nabla$, namely, $R_a(\tilde{L}) = \nabla(a - \tilde{L})$, is bounded in $L^p(M, \mathrm{e}^{-u}\mu)$ for all $a > 0$ and $p \geqslant 2$ provided that for some $B \geqslant 0$ and $\varepsilon > 0$,

$$\int_M \left[\left(\lambda_{\min}\left(\nabla^2 u(x) + \frac{\nabla\phi(x) \otimes \nabla\phi(x)}{m-n}\right) - C\rho^2(x) + B\right)^-\right]^{l/2+\varepsilon} \mathrm{d}\mu(x) < +\infty,$$

where

$$\lambda_{\min}\left(\nabla u^2(x) + \frac{\nabla\phi(x) \otimes \nabla\phi(x)}{m-n}\right)$$

denotes the lowest eigenvalue of the symmetric 2-tensor $\nabla^2 u + \frac{\nabla\phi \otimes \nabla\phi}{m-n}$ on $T_x M$, $\forall x \in M$.

8.5. L^p-Liouville theorem and L^p-uniqueness

For all $p \in (1, \infty)$, Yau [72] proved that the L^p-Liouville theorem holds for solutions of $Lu = 0$ on any complete non-compact Riemannian manifold. By Strichartz [65], it is well-known that the Laplace–Beltrami operator Δ is essentially self-adjoint in $L^2(M, \mathrm{d}x)$ and the heat semigroup $\mathrm{e}^{t\Delta}$ is $L^p(\mathrm{d}x)$-unique for all $p > 1$. By Bakry [7], the diffusion operator $L = \Delta - \nabla\phi \cdot \nabla$ is essentially self-adjoint on $L^2(M, \mu)$ for all $\phi \in C^2(M)$. By Sturm [66], it is known that for any complete non-compact Riemannian manifold M and for all $\phi \in C^2(M)$, the L^p-Liouville theorem holds for L-harmonic functions and the L^p-uniqueness holds for the heat semigroup $P_t = \mathrm{e}^{tL}$ for all $p \in (1, \infty)$. By the correspondence between the L^p-uniqueness of the heat semigroup $P_t = \mathrm{e}^{tL}$ and the one of the Schrödinger semigroup $Q_t = \mathrm{e}^{tH}$ generated by $H = \Delta - (|\nabla\phi|^2/4 - \Delta\phi/2)$, we conclude that $Q_t = \mathrm{e}^{tH}$ is unique in $L^p(M, \mathrm{e}^{(p-2)\phi/2}\,\mathrm{d}x)$ for all $p \in (1, \infty)$.

8.6. Infinite-dimensional case: open problems

The proofs of the main results of this paper are strongly relied on the generalized Laplace comparison theorem for diffusion operator with curvature-dimension condition. It seems to the author that one cannot extend the main results of this paper to infinite dimensional case if we want to keep the lower bound of the Bakry–Emery Ricci curvature to be a negative quadratic polynomial of the distance function. That is to say, the author cannot prove that the main results of this paper remain true on infinite dimensional manifolds under the negative quadratic curvature condition $Ric(x) + \nabla^2\phi(x) \geqslant -C(1 + d^2(x, o))$, $\forall x \in M$, where $d(x, o)$ denotes the Carnot–Carathéodory metric corresponding to L. Here, we would like to mention a result due to Bakry [6] which says that if $Ric(L) = Ric + \nabla^2\phi$ is uniformly bounded below by a negative constant, then the heat semigroup $P_t = \mathrm{e}^{tL}$ is conservative. Moreover, we refer the reader to Cruzeiro and Malliavin [19] and Bogachev, Röckner and Wang [14] and the reference therein for the study of existence, uniqueness and regularity of invariant measures for diffusion operators (which are not necessarily to be symmetric) on finite or infinite-dimensional manifolds under the strict positivity assumption of the Bakry–Emery Ricci curvature.

8.7. Liouville theorems for non-symmetric diffusion operators

As we have mentioned in Section 1.7, Qian [57] has studied the problems of conservation and the C_0-property for diffusion operators of the form $L = \Delta + B$ where B is not necessarily to be the gradient of some C^2-smooth function. Moreover, Bakry and Qian [13] proved that the generalized Laplace comparison theorem and the differential volume comparison theorem (for the L-invariant measure) hold for non-symmetric diffusion operator $L = \Delta + B$. To do this, they replaced $Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}$ by $Ric^S_{m,n}(L) = Ric - \nabla^S B - \frac{B\otimes B}{m-n}$, where $\nabla^S B$ is the symmetric part of ∇B defined by $\nabla^S B(\xi,\eta) = \frac{1}{2}\{\langle\nabla_\xi B,\eta\rangle + \langle\nabla_\eta B,\xi\rangle\}$, $\forall \xi,\eta \in TM$. See also Bakry and Qian [12]. In this case, the measure $\mathrm{d}\mu(x) = \mathrm{e}^{-\phi(x)}\,\mathrm{d}x$ should be replaced by an L-invariant measure which is a solution of the elliptic equation $L^*\mu = 0$, where L^* is given by $L^* f = \Delta f - \mathrm{div}(fB)$ for all $f \in C_0^\infty(M)$, see Ikeda and Watanabe [37] (p. 293) or Cruzeiro and Malliavin [19] and Bogachev, Röckner and Wang [14]. Due to these observations, it would be very possible to extend the main results in this paper to non-symmetric diffusion operators. We will study these problems in a forthcoming paper.

9. Generalized Calabi–Yau volume growth theorem and a criterion for the finiteness of the total mass of the invariant measure

9.1. Problem and background

In his report on the first submitted version of this paper, Professor P. Malliavin raised the following problem to the author.

Problem 9.1. What is the optimal geometric and analytic condition on M and ϕ such that the total mass of $\mu(\mathrm{d}x) = \mathrm{e}^{-\phi(x)}\,\mathrm{d}x$ is finite?

The well-known Bonnet–Myers theorem says that if M is a complete Riemannian manifold with Ricci curvature bounded below by a (strictly) positive constant, then M must be compact and hence has finite volume measure. Similarly, it has been known in the literature (cf. [9,19]) that if the (finite- or infinite-dimensional) Bakry–Emery Ricci curvature of the diffusion operator $L = \Delta - \nabla\phi\cdot\nabla$ is bounded from below by a (strictly) positive constant, that is, $Ric_{m,n}(L) = Ric + \nabla\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n} \geqslant K$, where $K > 0$ is a constant, $m = \infty$ or $m > n$ is a constant, then μ is a finite measure on M, that is, $\mu(M) = \int_M \mathrm{e}^{-\phi(x)}\,\mathrm{d}x < +\infty$. For example, the invariant measure of the Ornstein–Uhlenbeck operator $L = \Delta - x\cdot\nabla$ on Euclidean spaces or on the Wiener space is a finite measure since the infinite-dimensional Bakry–Emery Ricci curvature of L is identically equal to 1, that is, $Ric(L) = \mathrm{Id}_H$, where H is the Cameron–Martin subspace of the Wiener space.

9.2. The generalized Calabi–Yau volume growth theorem

A famous theorem due to Calabi [15] and Yau [73] says that if M is a complete non-compact Riemannian manifold with non-negative Ricci curvature, then M must have infinite volume. In general, we can extend Calabi–Yau's theorem to the following

Theorem 9.1. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $m \in (n, \infty)$ such that*

$$Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geqslant 0.$$

Then for all $o \in M$ and all $R > 0$, $\varepsilon > 0$, we have

$$\mu\big(B(o, R+3\varepsilon)\big) \geqslant \frac{R}{4m\varepsilon}\mu\big(B(o,\varepsilon)\big). \tag{9.49}$$

In additional, if M is non-compact, then

$$\mu(M) = \int_M \mathrm{e}^{-\phi(x)}\,\mathrm{d}x = +\infty.$$

Proof. By Remark 3.2, under the condition $Ric_{m,n}(L) \geqslant 0$, it holds that $L\rho^2 = 2\rho L\rho + 2 \leqslant 2m$ in the sense of distribution. That is, for all $\psi \in C_0^\infty(M, \mathbb{R}^+)$, we have

$$\int_M \psi L\rho^2\,\mathrm{d}\mu \leqslant 2m \int_M \psi\,\mathrm{d}\mu. \tag{9.50}$$

Since $C_0^\infty(M)$ is dense in $\mathrm{Lip}_0(M)$ which is the set of all Lipschitz functions with compact support in M, the inequality (9.50) still holds for all $\psi \in \mathrm{Lip}_0(M)$.

The rest of the proof is similar to the one of Calabi–Yau's theorem given in Schoen–Yau [63]. We fix a point $x_0 \in \partial B(o, R)$ and denote $\rho(x) = \mathrm{dist}(x_0, x)$ for all $x \in M$. Choosing $\psi(x) = u(\rho(x))$, where $u(t) = 1$ on $[0, R-\varepsilon]$, $u(t) = 0$ on $[R+\varepsilon, \infty)$ and satisfies $u'(t) = -\frac{1}{2\varepsilon}$ on $[R-\varepsilon, R+\varepsilon]$, then $\operatorname{supp}\psi \subset B(x_0, R+\varepsilon)$. Integrating by parts shows that

$$\begin{aligned}
\int_M \psi(x) L\rho^2(x)\,\mathrm{d}\mu(x) &= -\int_{B(x_0,R+\varepsilon)} \nabla\psi\cdot\nabla\rho^2\,\mathrm{d}\mu = -2\int_{B(x_0,R+\varepsilon)} u'\big(\rho(x)\big)\rho|\nabla\rho|^2\,\mathrm{d}\mu \\
&= \varepsilon^{-1} \int_{B(x_0,R+\varepsilon)\setminus B(x_0,R-\varepsilon)} \rho\,\mathrm{d}\mu \geqslant \varepsilon^{-1}(R-\varepsilon)\mu\big(B(x_0,R+\varepsilon)\setminus B(x_0,R-\varepsilon)\big).
\end{aligned}$$

On the other hand, the inequality (9.50) implies

$$\begin{aligned}
\int_{B(x_0,R+\varepsilon)\setminus B(x_0,R-\varepsilon)} \psi L\rho^2\,\mathrm{d}\mu &\leqslant 2m\int_M \psi\,\mathrm{d}\mu = 2m \int_{B(x_0,R+\varepsilon)} \psi\,\mathrm{d}\mu \\
&\leqslant 2m \int_{B(x_0,R+\varepsilon)} 1\,\mathrm{d}\mu = 2m\mu\big(B(x_0,R+\varepsilon)\big).
\end{aligned}$$

Note that $B(o,\varepsilon) \subset B(x_0, R+\varepsilon) \setminus B(x_0, R-\varepsilon)$. Hence

$$\begin{aligned} 2m\mu\big(B(x_0, R+\varepsilon)\big) &\geqslant \varepsilon^{-1}(R-\varepsilon)\mu\big(B(x_0, R+\varepsilon) \setminus B(x_0, R-\varepsilon)\big) \\ &\geqslant \varepsilon^{-1}(R-\varepsilon)\mu\big(B(o,\varepsilon)\big). \end{aligned}$$

On the other hand, $B(x_0, R+\varepsilon) \subset B(o, 2R+\varepsilon)$. Hence

$$\mu\big(B(o, 2R+\varepsilon)\big) \geqslant \frac{R-\varepsilon}{2m\varepsilon}\mu\big(B(o,\varepsilon)\big), \quad \forall R > \varepsilon > 0.$$

Replacing R by $(R+2\varepsilon)/2$, we have

$$\mu\big(B(o, R+3\varepsilon)\big) \geqslant \frac{R}{4m\varepsilon}\mu\big(B(o,\varepsilon)\big), \quad \forall R > 0,\ \varepsilon > 0.$$

The proof of theorem is completed. □

Remark 9.1. Under the same condition as in Theorem 9.1, the generalized Laplacian comparison theorem implies that $\mu(B(o,R)) \leqslant \mu(B(o,\varepsilon))(R/\varepsilon)^m$. To see this, we need only to take $K(R) \equiv 0$ in Lemma 3.1. Combining this with (9.49), we have the following two-sides volume growth comparison inequality for any L-invariant measure μ on complete Riemannian manifolds with $Ric_{m,n}(L) \geqslant 0$, namely,

$$\frac{(R-3\varepsilon)^+}{4m\varepsilon}\mu\big(B(o,\varepsilon)\big) \leqslant \mu\big(B(o,R)\big) \leqslant \mu\big(B(o,\varepsilon)\big)\left(\frac{R}{\varepsilon}\right)^m, \quad \forall R > \varepsilon > 0.$$

9.3. A upper bound diameter estimate

As a corollary of the estimate (9.49), we have the following result for the compactness of complete Riemannian manifolds which seems new in the literature. We refer the reader to Bakry and Ledoux [11] for another type of diameter upper bound estimate in terms of the mean of the distance function on compact Riemannian manifolds.

Theorem 9.2. *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that there exists a constant $m \in (n, \infty)$ such that*

$$Ric_{m,n}(L) = Ric + \nabla^2\phi - \frac{\nabla\phi \otimes \nabla\phi}{m-n} \geqslant 0.$$

If $\mu(M) = \int_M e^{-\phi(x)}\,dx < +\infty$, then M is compact. Moreover, for any $\varepsilon > 0$ small enough, the diameter of M satisfies

$$D(M) \leqslant \frac{8m\varepsilon\mu(M)}{\sup_{o\in M}\mu(B(o,\varepsilon))}. \tag{9.51}$$

Remark 9.2. After the author proved the above diameter estimate (9.51), we are aware from M. Ledoux that Lott and Villani [48] have recently proved the so-called weak Myers theorem on metric-measure spaces via a combination of the Bishop type volume inequality and the Talagrand optimal transport inequality. Their result says that if the $CD(0,m)$ and the $CD(K,\infty)$ conditions hold on a compact measured length space, where $K>0$ is a constant, then the diameter of M satisfies $D(M)\leqslant C\sqrt{m/K}$ for some constant $C>0$ independent of m and K. As mentioned in the beginning of this section, $CD(K,\infty)$ with $K>0$ implies automatically $\mu(M)<\infty$, hence Theorem 9.2 and our diameter estimate (9.51) apply on complete Riemannian manifolds equipped with a L-invariant measure satisfying the $CD(0,m)$ and the $CD(K,\infty)$ conditions for some $m>n$ and $K>0$.

9.4. A criterion of the finiteness of the total mass of the L-invariant measure

We now turn to Problem 9.1 posed by Professor P. Malliavin. The following theorem gives a criterion for the finiteness of the total mass of the L-invariant measure on complete non-compact Riemannian manifolds. We owe this criterion to D. Bakry with whom a stimulating discussion leads us to obtain the result.

Theorem 9.3. *Let M be a complete non-compact Riemannian manifold, $\phi\in C^2(M)$. Suppose that there exist a fixed point $o\in M$, a constant $m\in[n,\infty)$ and a function K on $\mathbb{R}^+$ such that for all $x\in M$,*

$$Ric_{m,n}(L)=Ric+\nabla^2\phi-\frac{\nabla\phi\otimes\nabla\phi}{m-n}\geqslant K\big(\rho(x)\big),$$

where $m=n$ if and only if ϕ is identically equal to a constant. Then

$$\mu(M)=\int_M \mathrm{e}^{-\phi(x)}\,\mathrm{d}x<+\infty$$

provided that for some $\varepsilon>0$,

$$\mu_K\big([\varepsilon,+\infty)\big):=\int_\varepsilon^\infty \mathrm{e}^{\int_\varepsilon^r a_K(s)\,\mathrm{d}s}\,\mathrm{d}r<+\infty,$$

where a_K is the solution to the Riccati equation

$$-a_K'=K+\frac{a_K^2}{m-1}\quad \textit{in } \mathbb{R}^+\setminus\{0\}$$

with the boundary condition $\lim_{r\to 0+} ra_K(r)=m-1$.

Notice that by the extended Myers theorem (see Part (1) in Theorem 1.1), since M is a complete non-compact Riemannian manifold, the solution a_K to the above Riccati equation is globally well defined on $(0,\infty)$. Let

$$L_K = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_K(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

be the uni-dimensional diffusion operator on $(0,\infty)$ with an invariant measure given by

$$\mu_K(\mathrm{d}x) = \mathrm{e}^{-\int_\varepsilon^x a_K(y)\,\mathrm{d}y}\,\mathrm{d}x, \quad \forall \varepsilon > 0.$$

Similarly to the role of the space forms (which are complete and simple connected Riemannian manifolds with constant sectional curvature) in Riemannian geometry, $((0,\infty), L_K, \mu_K)$ (in fact $([\varepsilon,\infty), L_K, \mu_K)$) provides us with the model for the comparison with (M, L, μ).

9.5. Proofs of Theorem 9.3

We would like to give three different proofs of Theorem 9.3.

Proof I. Let $\mathrm{d}\mu_\sigma(x)$ be the area measure on $\partial B(o,r)$. Using the co-area formula and the generalized Laplacian comparison theorem, we have

$$\begin{aligned}
\int\limits_{B(o,R)\setminus B(o,r)} L\rho(x)\,\mathrm{d}\mu(x) &= \int\limits_r^R \int\limits_{\partial B(o,s)} L\rho(x)\,\mathrm{d}\mu_\sigma(x)\,\mathrm{d}s \\
&\leqslant \int\limits_r^R \int\limits_{\partial B(o,s)} a_K\big(\rho(x)\big)\,\mathrm{d}\mu_\sigma(x)\,\mathrm{d}s \\
&= \int\limits_r^R a_K(s)\mu_\sigma\big(\partial B(o,s)\big)\,\mathrm{d}s.
\end{aligned}$$

On the other hand, the Green formula yields

$$\begin{aligned}
\int\limits_{B(o,R)\setminus B(o,r)} L\rho(x)\,\mathrm{d}\mu(x) &= \int\limits_{\partial(B(o,R)\setminus B(o,r))} \frac{\partial\rho(x)}{\partial\nu}\,\mathrm{d}\mu_\sigma(x) \\
&= \mu_\sigma\big(\partial B(o,R)\big) - \mu_\sigma\big(\partial B(o,r)\big).
\end{aligned}$$

Combining the above formulae with

$$\mu_\sigma\big(\partial B(o,r)\big) = \frac{\partial\mu(B(o,r))}{\partial r}$$

and denoting $V_R = \mu(B(o, R))$, we obtain

$$V'_R - V'_r \leqslant \int_r^R a_K(s) V'_s \, \mathrm{d}s, \quad \forall R > r > 0.$$

This yields that

$$V''_r \leqslant a_K(r) V'_r, \quad \forall r > 0.$$

Then it is easy to verify that

$$pV_R \leqslant V_r + V'_r \int_r^R \mathrm{e}^{\int_r^s a_K(\tau)\,\mathrm{d}\tau} \, \mathrm{d}s, \quad \forall R > r > 0.$$

Letting $R \to \infty$, we have

$$\mu(M) \leqslant V_r + V'_r \int_r^\infty \mathrm{e}^{\int_r^s a_K(\tau)\,\mathrm{d}\tau} \, \mathrm{d}s, \quad \forall r > 0. \qquad \square$$

Proof II. Similarly to the proof of Lemma 3.1, integrating $L\rho^2$ on $B(o, R) \setminus B(o, r)$, where $R > r > 0$, and applying the Green formula and the co-area formula, we can prove that

$$RV'_R - rV'_r \leqslant V_R - V_r + \int_{B(o,R)\setminus B(o,r)} \rho(x) L\rho(x) \, \mathrm{d}\mu(x),$$

where

$$V_r = \mu\big(B(o, r)\big), \quad V'_r = \frac{\partial \mu(B(o, r))}{\partial r}, \quad \forall r > 0.$$

Moreover, the generalized Laplacian comparison formula and the co-area formula imply:

$$\begin{aligned}
\int_{B(o,R)\setminus B(o,r)} \rho(x) L\rho(x) \, \mathrm{d}\mu(x) &\leqslant \int_{B(o,R)\setminus B(o,r)} \rho(x) a_K\big(\rho(x)\big) \, \mathrm{d}\mu(x) \\
&= \int_r^R s a_K(s) \mu_\sigma\big(\partial B(o, s)\big) \, \mathrm{d}s \\
&= \int_r^R s a_K(s) V'_s \, \mathrm{d}s.
\end{aligned}$$

Hence

$$RV_R' - rV_r' \leqslant V_R - V_r + \int_r^R s a_K(s) V_s' \,\mathrm{d}r, \quad \forall R > r > 0.$$

This yields that

$$(rV_r')' \leqslant V_r' + r a_K(r) V_r', \quad \forall r > 0.$$

That is

$$V_r'' \leqslant a_K(r) V_r', \quad \forall r > 0.$$

This implies the desired result in Theorem 9.3. □

Proof III. Choosing the normal polar coordinate system (r,θ) near $o \in M$, where $r \in \mathbb{R}^+$, $\theta \in S^{n-1}$, the Gauss lemma implies that

$$\mathrm{d}s_M^2 = \mathrm{d}r^2 + g_{\alpha\beta}(r,\theta)\,\mathrm{d}\theta^\alpha\,\mathrm{d}\theta^\beta, \quad \alpha, \beta = 2, \ldots, n.$$

Letting

$$J(r,\theta) = \sqrt{\det\big(g_{\alpha\beta}(r,\theta)\big)}$$

be the area density of the geodesic sphere $\partial B(o,r)$, then

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial \log J}{\partial r}\frac{\partial}{\partial r} + \Delta_{\partial B(o,r)}.$$

Hence on $M \setminus cut(o)$, we have

$$Lr = \frac{\partial \log J}{\partial r} - \nabla\phi \cdot \nabla r = \frac{\partial \log J}{\partial r} - \frac{\partial \phi(r,\theta)}{\partial r}.$$

By the co-area formula, see also Lemma 6.2, the weight area measure of μ on $\partial B(o,r)$ is given by

$$\mathrm{d}\mu_\sigma(r,\theta) = \mathrm{e}^{-\phi(r,\theta)} J(r,\theta)\,\mathrm{d}\theta,$$

where $\mathrm{d}\theta$ denotes the standard area measure on S^{n-1}. Let

$$J_\phi(r,\theta) = \mathrm{e}^{-\phi(r,\theta)} J(r,\theta). \tag{9.52}$$

Then

$$Lr = \frac{\partial}{\partial r} \log J_\phi(r,\theta) \quad \text{on } M \setminus cut(o). \tag{9.53}$$

Therefore, the generalized Laplacian comparison theorem is equivalent to the differential inequality

$$J'_\phi(r,\theta) \leqslant a_K(r) J_\phi(r,\theta) \quad \text{on } M \setminus cut(o).$$

Combining this with

$$\mu\big(B(o,R)\big) = \int_0^R \int_{\partial B(o,r)} J_\phi(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta,$$

the standard argument yields that

$$\mu\big(B(o,R)\big) \leqslant \mu\big(B(o,r)\big) + \mu_\sigma\big(\partial B(o,r)\big) \int_r^s \mathrm{e}^{\int_r^s a_K(\tau)\,\mathrm{d}\tau}\,\mathrm{d}s, \quad \forall R > r > 0.$$

The proofs of Theorem 9.3 are completed. □

10. A variational approach to the Bakry–Qian Laplacian comparison theorem

In [13], Bakry and Qian proved the generalized Laplacian comparison theorem for the diffusion operator without using the Jacobi fields theory. To reader who is not familiar in the theory of Γ_2 which serves as an indispensable tool in [13], it might be useful to give a new proof for this result without using Γ_2. Our proof uses the variational formulae in Riemannian geometry and is accessible to people in geometric analysis.

Proof of Theorem 1.1. Choosing the normal polar coordinate system (r,θ) at $o \in M$, where $r \in \mathbb{R}^+$, $\theta \in S^{n-1}$, the Gauss lemma implies that

$$\mathrm{d}s_M^2 = \mathrm{d}r^2 + g_{\alpha\beta}(r,\theta)\,\mathrm{d}\theta^\alpha\,\mathrm{d}\theta^\beta, \quad \alpha,\beta = 2,\ldots,n.$$

Let

$$J(r,\theta) = \sqrt{\det\big(g_{\alpha\beta}(r,\theta)\big)},$$
$$J_\phi(r,\theta) = \mathrm{e}^{-\phi(r,\theta)} J(r,\theta).$$

Then

$$\mathrm{d}\mu(x) = J_\phi(r,\theta)\,\mathrm{d}r\,\mathrm{d}\theta.$$

By the first and the second variational formulae in Riemannian geometry, for all $x = (r,\theta) \in M \setminus cut(o)$, we have

$$J'(r,\theta)=\frac{\partial J}{\partial r}(r,\theta)=H(r,\theta)J(r,\theta),$$

and

$$J''(r,\theta)=\frac{\partial^2 J}{\partial r^2}(r,\theta)=\left[-\sum_{i,j=1}^{n-1}h_{ij}^2(r,\theta)-Ric\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)+H^2(r,\theta)\right]J(r,\theta),$$

where $H(r,\theta)$ denotes the mean curvature at $x=(r,\theta)$, $(h_{ij}(r,\theta))$ denotes the second fundamental form of $\partial B(o,r)$ at $x=(r,\theta)$ with respect to the unit normal vector $\frac{\partial}{\partial r}$.

Denote $\phi'=\frac{\partial\phi}{\partial r}(r,\theta)$, $\phi''=\frac{\partial^2\phi}{\partial r^2}(r,\theta)$. Then

$$J_\phi'(r,\theta)=\frac{\partial J_\phi}{\partial r}(r,\theta)=[H-\phi']J_\phi(r,\theta),$$

$$J_\phi''(r,\theta)=\frac{\partial^2 J_\phi}{\partial r^2}(r,\theta)=[H'-\phi'']J_\phi(r,\theta)+[H-\phi']^2J_\phi(r,\theta). \tag{10.54}$$

Substituting

$$H'(r,\theta)=-\sum_{i,j=1}^{n-1}h_{ij}^2(r,\theta)-Ric\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)$$

into (10.54), we obtain

$$\begin{aligned}\frac{J_\phi''}{J_\phi}&=-\sum_{i,j=1}^{n-1}h_{ij}^2-Ric\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)-\phi''+H^2-2H\phi'+\phi'^2\\&=-Ric(L)\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)-\sum_{i,j=1}^{n-1}h_{ij}^2+H^2-2H\phi'+\phi'^2\\&=-Ric_{m,n}(L)\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)+\frac{m-n-1}{m-n}\phi'^2+H^2-2H\phi'-\sum_{i,j=1}^{n-1}h_{ij}^2.\end{aligned}$$

Notice that

$$\sum_{i,j=1}^{n-1}h_{ij}^2\geqslant\sum_{i=1}^{n-1}h_{ii}^2\geqslant\frac{(\sum_{i=1}^{n-1}h_{ii})^2}{n-1}=\frac{H^2}{n-1}.$$

Hence

$$\begin{aligned}
\frac{J_\phi''}{J_\phi} &\leqslant -Ric_{m,n}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)+\frac{m-n-1}{m-n}\phi'^2+\frac{n-2}{n-1}H^2-2H\phi' \\
&= \frac{m-2}{m-1}[H-\phi']^2-Ric_{m,n}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)+\frac{m-n-1}{m-n}\phi'^2-2H\phi'-\frac{m-2}{m-1}[H-\phi']^2 \\
&= \frac{m-2}{m-1}\left[\frac{J_\phi'}{J_\phi}\right]^2-Ric_{m,n}\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right)+\frac{m-n-1}{m-n}\phi'^2-2H\phi'-\frac{m-2}{m-1}[H-\phi']^2.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\frac{m-n-1}{m-n}\phi'^2-2H\phi'-\frac{m-2}{m-1}[H-\phi']^2 \\
&\quad= -\frac{n-1}{(m-n)(m-1)}\phi'^2-\frac{2}{m-1}H\phi'-\frac{m-n}{(m-1)(n-1)}H^2 \\
&\quad= -\frac{n-1}{(m-n)(m-1)}\left[\phi'+\frac{1/(m-1)}{(n-1)/(m-n)(m-1)}H\right]^2 \\
&\quad= -\frac{n-1}{(m-n)(m-1)}\left[\phi'+\frac{m-n}{n-1}H\right]^2 \\
&\quad\leqslant 0.
\end{aligned}$$

Therefore

$$\frac{J_\phi''}{J_\phi}\leqslant\frac{m-2}{m-1}\left[\frac{J_\phi'}{J_\phi}\right]^2-Ric_{m,n}(L)\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right).$$

Let $u=J_\phi'/J_\phi$. Then

$$u'=\frac{J_\phi''}{J_\phi}-\left[\frac{J_\phi'}{J_\phi}\right]^2\leqslant-\frac{1}{m-1}\left[\frac{J_\phi'}{J_\phi}\right]^2-Ric_{m,n}(L)\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right).$$

That is to say, on $M\setminus cut(o)$, we have the Riccati differential inequality

$$-u'\geqslant\frac{u^2}{m-1}+Ric_{m,n}(L)\left(\frac{\partial}{\partial r},\frac{\partial}{\partial r}\right). \tag{10.55}$$

Since any Riemannian metric on M is locally Euclidean, as $r\to 0^+$, we have:

$$J(r,\theta)\simeq r^{n-1}\quad\text{and}\quad J'(r,\theta)\simeq(n-1)r^{n-2}.$$

This implies that, as $r\to 0^+$,

$$J_\phi(r,\theta)\simeq \mathrm{e}^{-\phi(r,\theta)}r^{n-1},$$

and

$$J'(r,\theta) \simeq \left[-\phi'(r,\theta) + \frac{n-1}{r}\right] J_\phi(r,\theta).$$

Since $m \geqslant n$ (where $m = n$ if and only if ϕ is identically equal to a constant), we have

$$u(r,\theta) \simeq \left[-\phi'(r,\theta) + \frac{n-1}{r}\right] \leqslant \frac{m-1}{r}, \quad \text{as } r \to 0^+.$$

Hence, under the curvature condition $Ric_{m,n}(L) \geqslant K$, we have:

$$-u' \geqslant \frac{u^2}{m-1} + K$$

and the boundary condition

$$\lim_{r\to 0^+} ru(r,\theta) = n-1 \leqslant m-1.$$

The well-known Sturm–Liouville comparison theorem of the Riccati equation implies that

$$u \leqslant a_K \quad \text{on } M \setminus cut(o),$$

where a_K is the solution to the Riccati equation

$$-a_K' = \frac{a_K^2}{m-1} + K \tag{10.56}$$

with the boundary condition

$$\lim_{r\to 0^+} ra_K(r,\theta) = m-1. \tag{10.57}$$

Hence, the Bishop volume differential inequality holds

$$\frac{J_\phi'}{J_\phi}(r,\theta) \leqslant a_K(r) \quad \text{on } x = (r,\theta) \in M \setminus cut(o). \tag{10.58}$$

As in Proof III of Theorem 9.3, the Gauss lemma implies that

$$Lr = \frac{J_\phi'}{J_\phi} \quad \text{on } M \setminus cut(o).$$

Combining this with (10.58), we have

$$Lr \leqslant a_K(r) \quad \text{on } x = (r,\theta) \in M \setminus cut(o).$$

Moreover, this also implies that $\Delta r = Lr + \phi'$ can only be well-defined on $(0, \theta_K) \times S^{n-1}$, where δ_K denotes the explosion time of the solution a_K to the Riccati equation (10.56) with the initial condition (10.57) such that

$$\lim_{r \to \delta_K -} a_K(r) = -\infty.$$

Hence the Myers theorem holds, that is, the diameter of M is less than δ_K, i.e.,

$$D(M) \leqslant \delta_K.$$

The proof of Theorem 1.1 is completed. □

Acknowledgements

I am very grateful to Professor D. Bakry for stimulating and useful discussions during many stages of this paper, to Professor A. Grigor'yan for helpful discussions in the preparation, and to Professor P. Malliavin for valuable suggestions on the modification. Part of the work was done when I visited the Academy of Mathematics and System Sciences of the Chinese Academy of Sciences and the Departments of Mathematics of Zhongshan University and of Fudan University. I would like to thank Professors Z.-M. Ma, X.-P. Zhu and B.-L. Chen, S.-J. Tang and J.-G. Ying for their invitation, hospitality and helpful discussions. I also thank Professors T. Coulhon, T. Delmotte, F.-Z. Gong, E. Hsu, M. Ledoux, T. Lyons, Z.-M. Qian and L.-M. Wu for useful discussions. Parts of this work have been reported at the International Workshop on Markov Processes and Related Topics at Beijing Normal University, August 10–14, 2004. I am grateful to Professors M.-F. Chen, Z.-F. Li and Y.-H. Zhang for their invitation. Financial support from le Laboratoire de Statistique et Probabilités de l'Université Paul Sabatier for the international travel expense between France and China is gratefully acknowledged.

References

[1] A. Ancona, Negatively curved manifolds, elliptic operators, and the Martin boundary, Ann. of Math. 125 (1987) 495–536.
[2] M. Anderson, The Dirichlet problem at infinity for manifolds of negative curvature, J. Differential Geom. 18 (1983) 701–721.
[3] M. Anderson, R. Schoen, Positive harmonic functions on complete manifolds with negative curvature, Ann. of Math. 121 (1985) 429–461.
[4] D.G. Aronson, Uniqueness of positive weak solutions of second order parabolic equations, Ann. Polon. Math. XVI (1965) 286–303.
[5] R. Azencott, Behavior of diffusion semi-groups, Bull. Soc. Math. France 102 (1974) 193–240.
[6] D. Bakry, Un critère de non-explosion pour certaines diffusions sur une variété riemannienne complète, C. R. Acad. Sci. Paris, Sér. I 303 (1986) 23–26.
[7] D. Bakry, Étude des transformations de Riesz dans les variétés riemanniennes à courbure de Ricci minorée, in: Sém. Prob. XXI, in: Lecture Notes in Math., vol. 1247, Springer-Verlag, Berlin/New York, 1987, pp. 137–172.

[8] D. Bakry, L'hypercontractivité et son utilisation en théorie des semigroupes, in: Lecture Notes in Math., vol. 1581, Springer-Verlag, Berlin/New York, 1994, pp. 1–114.

[9] D. Bakry, Functional inequalities for Markov semigroups, Lecture Note in Tata Institute, Bombay, Preprint, 2002.

[10] D. Bakry, M. Emery, Diffusion hypercontractives, in: Sém. Prob. XIX, in: Lecture Notes in Math., vol. 1123, Springer-Verlag, Berlin/New York, 1985, pp. 177–206.

[11] D. Bakry, M. Ledoux, A logarithmic Sobolev form of the Li–Yau parabolic inequality, Rev. Mat. Iberoamericana, 2005, in press.

[12] D. Bakry, Z.-M. Qian, Harnack inequalities on a manifold with positive or negative Ricci curvature, Rev. Mat. Iberoamericana 15 (1999) 143–179.

[13] D. Bakry, Z.-M. Qian, Volume comparison theorems without Jacobi fields, Preprint 2003. Available at http://www.lsp.ups-tlse.fr/Bakry.

[14] V.I. Bogachev, M. Röckner, F.-Y. Wang, Elliptic equations for invariant measures on finite and infinite dimensional manifolds, J. Math. Pures Appl. 80 (2001) 177–221.

[15] E. Calabi, On manifolds with non-negative Ricci curvature II, Notices Amer. Math. Soc. 22 (1975) A205.

[16] S.Y. Cheng, P. Li, S.T. Yau, On the upper estimate of the heat kernel of a complete Riemannian manifold, Amer. J. Math. 103 (1981) 1021–1063.

[17] S.Y. Cheng, S.T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. 28 (1975) 333–354.

[18] L.O. Chung, Existence of harmonic L^1 functions in complete Riemannian manifolds, Proc. Amer. Math. Soc. 88 (1983) 531–532.

[19] A.B. Cruzeiro, P. Malliavin, Non perturbative construction of invariant measure through confinement by curvature, J. Math. Pures Appl. 77 (1998) 527–537.

[20] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Univ. Press, Cambridge, 1989.

[21] E.B. Davies, L^1 properties of second order elliptic operators, Bull. London Math. Soc. 17 (1985) 417–436.

[22] E.B. Davies, Heat kernel bounds, conservation of probability and the Feller property, J. Anal. Math. 58 (1992) 99–199.

[23] A. Debiard, B. Gaveau, E. Mazet, Théorèmes de comparaison en géométrie riemannienne, Publ. Res. Inst. Math. Sci. 12 (2) (1976/1977) 391–425.

[24] J. Dodziuk, Maximum principle for parabolic inequalities and the heat flow on open manifolds, Indiana Univ. Math. J. 32 (1983) 703–716.

[25] K.D. Elworthy, Stochastic flows and the C_0-diffusion property, Stochastics 6 (1982) 233–238.

[26] W. Feller, The parabolic differential equations and the associated semigroups of transformations, Ann. of Math. 55 (1952) 468–519.

[27] L. Garnett, Foliations, the ergodic theorem and Brownian motion, J. Funct. Anal. 51 (1983) 285–311.

[28] F.-Z. Gong, F.-Y. Wang, Heat kernel estimates with application to compactness of manifolds, Quart J. Math. Ser. 52 (2001) 1–10.

[29] R. Greene, H. Wu, Integrals of subharmonic functions on manifolds of nonnegative curvature, Invent. Math. 27 (1974) 265–298.

[30] A.A. Grigor'yan, On stochastically complete manifolds, DAN SSSR 290 (1986) 534–537 (in Russian). Engl. transl. Soviet Math. Dokl. 34 (1987) 310–313.

[31] A.A. Grigor'yan, Stochastically complete manifolds and summable harmonic functions, Izv. AN SSSR, Ser. Math. 52 (1988) 1102–1108 (in Russian). Engl. transl. Math. USSR Izvestiya 33 (1989) 425–432.

[32] A.A. Grigor'yan, The heat equation on non-compact Riemannian manifolds, Mat. Sbornik 182 (1991) 55–87 (in Russian). Engl. transl. Math. USSR Sb. 72 (1992) 47–77.

[33] A.A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. (N.S.) 36 (1999) 135–249.

[34] P. Hsu, P. March, The limiting angle of certain Riemannian Brownian motions, Commun. Pure Appl. Math. 38 (1985) 755–768.

[35] P. Hsu, Heat semigroup on a complete Riemannian manifold, Ann. Probab. 17 (1989) 1248–1254.

[36] E. Hsu, Stochastic Analysis on Manifolds, Grad. Stud. Math., vol. 38, Amer. Math. Soc., Providence, RI, 2002.

[37] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1984.

[38] W.S. Kendall, The radial part of Brownian motion on a manifold, A semimartingale property, Ann. Probab. 15 (1987) 1491–1500.
[39] L. Karp, P. Li, The heat equation on complete Riemannian manifolds, unpublished manuscript.
[40] A.N. Kolmogoroff, Zur Umkehrbarkeit der statischen Naturgesetze, Math. Ann. 113 (1937) 766–772; English translation can be found in: A.N. Shiryayev (Ed.), Selected Works of A.N. Kolmogorov, vol. II, Probability Theory and Mathematical Statistics, Kluwer Academic, Dordrecht/Norwell, MA, 1992.
[41] P. Li, Uniqueness of L^1 solutions for the Laplace equation and the heat equation on Riemannian manifolds, J. Differential Geom. 20 (1984) 447–457.
[42] P. Li, R. Schoen, L^p and mean value properties of subharmonic functions on Riemannian manifolds, Acta Math. 153 (1984) 279–301.
[43] P. Li, S.T. Yau, On the parabolic kernel of the Schrödinger operator, Acta. Math. 156 (1986) 153–201.
[44] X.D. Li, Riesz transforms for symmetric diffusion operators on complete Riemannian manifolds, Rev. Mat. Iberamericana, 2005, in press.
[45] X.-M. Li, Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds, Probab. Theory and Related Fields 104 (1994) 485–511.
[46] V. Liskevitch, On the uniqueness problem for Dirichlet operators, J. Funct. Anal. 162 (1999) 1–13.
[47] J. Lott, Some geometric properties of the Bakry–Emery Ricci tensor, Comment. Math. Helv. 78 (2003) 865–883.
[48] J. Lott, C. Villani, Ricci curvature for metric-measured spaces via optimal transport, Preprint, January 30, 2005.
[49] T. Lyons, Instability of the Liouville property for quasi-isometric Riemannian manifolds and reversible Markov chain, J. Differential Geom. 26 (1987) 33–66.
[50] T. Lyons, Instability of the conservative property under quasi-isometries, J. Differential Geom. 34 (1991) 483–489.
[51] P. Malliavin, Asymptotic of the Green's function of a Riemannian manifold and Itô's stochastic integral, Proc. Nat. Acad. Sci. 71 (1974) 381–383.
[52] P. Malliavin, Formule de la moyenne, calcul des perturbations et théorie d'annulation pour les formes harmoniques, J. Funct. Anal. 17 (1974) 274–291.
[53] P. Malliavin, Diffusions et géométrie différentielle globale, Centro Matem. Estivo Varenna (1975) 208–279.
[54] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, http://arXiv.org/abs/maths/0211159.
[55] R.G. Pinsky, Positive Harmonic Functions and Diffusion, An Integrated Analytic and Probabilistic Approach, Cambridge Stud. Adv. Math., vol. 45, Cambridge Univ. Press, Cambridge, UK, 1995.
[56] J.-J. Prat, Étude asymptotique et convergence angulaire du mouvement brownien sur une variété à courbure négative, C. R. Acad. Sci. Paris Sér. A–B 280 (1975) A1539–A1542.
[57] Z.-M. Qian, On conservation of probability and the Feller property, Ann. Probab. 24 (1996) 280–292.
[58] Z.-M. Qian, Estimates for weight volumes and applications, J. Math. Oxford Ser. 48 (1987) 235–242.
[59] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Differential Geom. 36 (1992) 417–450.
[60] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, Internat. Math. Res. Notices 2 (1992) 27–38.
[61] L. Saloff-Coste, Aspects of Sobolev-Type Inequalities, London Math. Soc. Lecture Note Ser., vol. 289, Cambridge Univ. Press, Cambridge, UK, 2002.
[62] R. Schoen, The effect of curvature on the behavior of harmonic functions and mappings, in: R. Hardt, M.W. Wolf (Eds.), Nonlinear Partial Differential Equations in Differential Geometry, in: IAS/Park City Math. Series, vol. 2, Amer. Math. Soc., Institute for Advanced Study, 1996, pp. 127–184.
[63] R. Schoen, S.T. Yau, Lectures on Differential Geometry, International Press, Cambridge, MA, 1994.
[64] W. Stannat, (Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 28 (1999) 99–140.
[65] R. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold, J. Funct. Anal. 52 (1983) 48–79.
[66] K.T. Sturm, Analysis on local Dirichlet spaces I. Recurrence, conservativeness and L^p-Liouville properties, J. Reine Angew. Math. 456 (1994) 173–196.

[67] D. Sullivan, The Dirichlet problem at infinity for a negative curved manifold, J. Differential Geom. 18 (1983) 723–732.

[68] N.Th. Varopoulos, Potential theory and diffusion on Riemannian manifolds, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund, vols. I, II, in: Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 821–837.

[69] N.Th. Varopoulos, Small time Gaussian estimates of heat diffusion kernels, Part I: The semigroup technique, Bull. Soc. Math. 2^{e} Sér. 113 (1989) 253–277.

[70] J. Vauthier, Théorèmes d'annulation et de finitude d'espaces de 1-formes harmoniques sur une variété de Riemann ouverte, Bull. Sci. Math. (2) 103 (2) (1979) 129–177.

[71] L.-M. Wu, Uniqueness of Nelson's diffusions, Probab. Theory and Related Fields 114 (1999) 549–585.

[72] S.T. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975) 201–228.

[73] S.T. Yau, Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry, Indiana Math. J. 25 (1976) 659–670.

[74] S.T. Yau, On the heat kernel of a complete Riemannian manifold, J. Math. Pures Appl. (9) 57 (2) (1978) 191–201.